# A Homotopical Categorification of the Euler Calculus

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## Abstract

Euler calculus is an analogue of the theory of integration for constructible functions rather than measurable ones. Due to its computationally accessible nature, Euler calculus plays a central role in aspects of applied algebraic topology, for example in enumeration problems involving networks of sensors. A geometric description of the constructible functions is given by the Grothendieck group of the constructible derived category. This sheaf-theoretic categorification of the constructible functions is well-known. We present an alternative geometric categorification of the constructible functions given by a suitable homotopy category; an analogue of the classical Spanier–Whitehead category but for suitably 'tame' spaces over the source space of the constructible functions. To do so, we develop an axiomatic method for constructing Spanier–Whitehead categories given some ambient category with certain basic properties. The lifting of the operations of the Euler calculus to functors between these Spanier–Whitehead categories should illuminate homotopical aspects of the Euler calculus.

In memory of my father,
Anthony Austin Moggach,
1946 – 2015,
who hoped so much to ride the Mersey ferry.

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# Contents

# Introduction

Euler calculus was developed in the late 1980s by Viro [Vir88] and by Schapira [Sch91, Sch95]. It provides an integration theory for constructible functions which allows one to study the topology of constructible sets and functions. Viro started from the observation that compactly supported Euler characteristic $\chi_c$, is additive and so is almost a measure, the only difference being that it is not necessarily positive. From this perspective he developed the Euler integral by analogy with integration with respect to a measure. His main applications were to complex geometry and singularity theory — [GZ10] is a survey of this circle of ideas and its more recent relations with motivic measure and other topics in algebraic geometry. Schapira started from the fact that (under suitable conditions) constructible functions are the Grothendieck group of the constructible derived category of sheaves. The operations of the Euler calculus then arise as 'de-categorifications' of well-known operations on constructible sheaves. His applications were mainly in real analytic geometry, particularly to tomography and questions initiated from robotics. The survey paper [CGR12] focusses on yet other applications to sensing which have been developed by Baryshnikov, Ghrist and others; it also contains an extensive bibliography. The main objective of this thesis is to provide an alternative geometric categorification of the constructible functions via an appropriate homotopy category, and lift the operations of the Euler calculus to the underlying triangulated category of this homotopy category.

A calculus is a collection of rules for computation. The rules of the Euler calculus can be summarised as follows. The bounded constructible functions on a compact definable space $X$ form a ring $CF(X)$ which is generated by indicator functions of definable subsets. A continuous definable map $\beta\colon X \to Y$ induces functorial homomorphisms of abelian groups

$$\beta_*\colon CF(X) \to CF(Y) \quad \text{and} \quad \beta^*\colon CF(Y) \to CF(X)\,,$$

moreover $\beta^*$ is a ring homomorphism. Here, by 'functorial' we mean that $\beta_*\psi_* = (\beta\psi)_*$ and $\psi^*\beta^* = (\beta\psi)^*$ where $\beta$ and $\psi$ are composable definable maps. These satisfy, and are determined by,

(a) $\beta_*(1_A) = \chi(A)$ where $\chi$ is the Euler characteristic, $1_A$ is the indicator function of definable $A \subset X$ and $\beta\colon X \to \mathrm{pt}$ is the unique map to a point;

(b) $\beta^*(f) = f \circ \beta$;

(c) (base change) $b^* a_* = \alpha_* \beta^*$ whenever

$$\begin{tikzcd} W \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d, "b"] \\ Y \arrow[r, "a"'] & Z \end{tikzcd}$$

is a cartesian diagram;

(d) (projection formula) $\beta_*(f \cdot \beta^* g) = \beta_* f \cdot g$;

The key properties of the Euler calculus are captured by the list above. Following [Vir88, Sch91] we use the notation

$$\beta_*(f) = \int_X f \, d\chi$$

when $\beta \colon X \to \mathrm{pt}$ is the map to a point, and refer to pushforward to a point as taking the *Euler integral* or *integral with respect to the Euler characteristic* of the constructible function $f$.

The Euler characteristic is a topological invariant which is well-defined for spaces that have finite cell decompositions of some form. Accordingly, we want to focus on a suitably nice class of spaces which are well-behaved and have a natural cell decomposition into a finite number of cells. The *tame* spaces that we want to consider are the *definable* spaces. A space is said to be definable if it belongs to some o-minimal structure on $\mathbb{R}$ and is given the subspace topology of the usual Euclidean topology on $\mathbb{R}^m$.

An o-minimal structure $\mathcal{R} = \{\mathcal{R}_n\}$ on $\mathbb{R}$ is a collection of subsets of $\mathbb{R}^n$ for each $n \in \mathbb{N}$ such that

**S.1** $\mathcal{R}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$;

**S.2** $\mathcal{R}$ is closed under cartesian products;

**S.3** $\mathcal{R}_n$ contains the diagonals $\{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_i = x_j\}$ for any $i < j$;

**S.4** $\mathcal{R}$ is closed under projections $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ onto the first $n-1$ coordinates;

**O.1** $\mathcal{R}_2$ contains the subdiagonal $\{(x, y) \in \mathbb{R}^2 : x < y\}$;

**O.2** $\mathcal{R}_1$ consists of the finite unions of open intervals and points.

The first four axioms define the notion of a *structure*, and the final two axioms guarantee the *o-minimality* of the structure. Logicians are at the origin of o-minimal structures. However, it was in the 1980s that o-minimal structures began to be studied geometrically as a generalisation of classes such as the *semianalytic* and *semialgebraic* sets. In particular by the mathematicians Pillay and Steinhorn in [PS84] where the term “o-minimal structure” was used to highlight similarities with the model theoretic notion of a “strongly minimal structure”. The term o-minimal is short for “order minimal.” Over the last thirty years the subject has continued to grow, spurred on by a proof that the real exponential field is o-minimal provided by Wilkie in 1991 in [Wil96].

An o-minimal expansion of $\mathbb{R}$, which is an o-minimal structure on $\mathbb{R}$ that also contains the graphs of addition and multiplication, was first defined by van den Dries in [vdD84] and, in his own words, is ‘an excellent framework for developing tame topology, or *topologie modérée*,’ as laid out in Grothendieck’s “Esquisse d’un Programme” of 1984, [Gro97]. We set definability to mean definable in some fixed o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$. One of the key theorems on definable spaces is the Cell Decomposition Theorem, which tells us that nice cell decompositions of definable spaces exist (where the closure of any cell is a union of cells). It is important to realise that the notion of ‘cell decomposition’ in an o-minimal expansion of $\mathbb{R}$ is not the same as a CW-structure; however, Theorem 1.2.16 will relate these two concepts (as will be discussed shortly). In particular and as called for, a definable space has a well-defined Euler characteristic which is independent of the chosen cell decomposition. Another key theorem, known as the Trivialisation Theorem, says that one can find a cell decomposition of the codomain of a continuous definable map which is compatible with the decomposition of the domain (where a map is said to be definable if its graph is). There is also a definable Triangulation Theorem. The book [vdD98] by van den Dries is a very clear and complete reference for the study of o-minimal theory and tame topology. Alternatively, a concise but useful survey can be found in [Cos00].

We define **Def** to be the *definable category* with objects the compact definable spaces and morphisms the continuous definable maps. The pointed version (obtained by taking the coslice category of **Def** under the point $*$) is denoted by $\mathbf{Def}_*$ and has objects the pointed compact definable spaces and morphisms the basepoint-preserving continuous definable maps. A relative version of the definable category, where objects are both over and under some fixed object $Z \in \mathbf{Def}$, can be defined by taking the slice-coslice category, $\mathbf{Def}_Z := Z\backslash(\mathbf{Def}/Z)$, relative to the object $Z$. The *relative definable category* $\mathbf{Def}_Z$ will be particularly important in the last chapter of this thesis. We remark here that, in general, definable maps are not necessarily continuous functions. However, we will focus our attention on *continuous definable* maps.

The homotopy category that we use in order to categorify the constructible functions, and hence the Euler calculus, is a version of the Spanier–Whitehead category, as defined by Spanier and Whitehead in [SW62]. Although their original category had objects the pairs $(X, m)$ where $X$ is a pointed CW-complex, and morphisms from $(X, m)$ and $(Y, n)$ which are colimits over homotopy classes of continuous functions between suitably high suspensions of $\Sigma^m(X)$ and $\Sigma^n(Y)$, the more common and useful definition of the Spanier–Whitehead category $\mathbf{SW}(\mathbf{CW}_*)$ is the one in which the pointed CW-complexes are taken to be finite, as in for example [Sch10]. From Freudenthal's Suspension Theorem, [Fre38], since the colimit defining the morphisms in $\mathbf{SW}(\mathbf{CW}_*)$ is obtained by an iteration of suspensions, it must actually be attained at some finite stage. Thus, two finite pointed CW-complexes become isomorphic in $\mathbf{SW}(\mathbf{CW}_*)$ if and only if they become homotopy equivalent after some finite number of suspensions, and morphisms in $\mathbf{SW}(\mathbf{CW}_*)$ can be thought of as stabilised versions of homotopy classes. Formal de-suspensions exist in $\mathbf{SW}(\mathbf{CW}_*)$ due to the fact that the shift (functor) is isomorphic to the reduced suspension (functor). The Spanier–Whitehead category has an additive structure which arises from the morphism sets which are naturally abelian groups since every object can be written as a double suspension.

Crucially, $\mathbf{SW}(\mathbf{CW}_*)$ is a triangulated category. The triangles are given by the following mapping cone sequences

$$X \xrightarrow{f} Y \to C(f) \to \Sigma(X),$$

together with their suspensions and (de)-suspensions. The standard reference for the verification of the axioms of a triangulated category is Chapter 1 of Margolis' book [Mar83]. We note that Margolis uses the original Spanier–Whitehead category construction without imposing that objects be finite (as in [SW62]), and that $\mathbf{SW}_f$ is used to denote the finite version of the category. Also, this verification does not include the octahedral axiom. In fact, a definitive discussion on the subject which does explicitly verify the octahedral axiom is difficult to come by. We will follow Margolis' survey in [Mar83] as a reference for the classical Spanier–Whitehead category.

The two analogues of the Spanier–Whitehead category that we are interested in are the definable Spanier–Whitehead category $\mathbf{SW}(\mathbf{Def}_*)$, and the relative definable Spanier–Whitehead category $\mathbf{SW}(\mathbf{Def}_Z)$. Both categories will be triangulated, and the latter category will be the one used to categorify the constructible functions (on $Z$). In order to achieve this categorification, we study the Grothendieck groups $K(\mathbf{SW}(\mathbf{Def}_*))$ and $K(\mathbf{SW}(\mathbf{Def}_Z))$ and prove that there exist ring isomorphisms

$$K(\mathbf{SW}(\mathbf{Def}_*)) \cong \mathbb{Z}, \tag{0.0.1}$$

and

$$K(\mathbf{SW}(\mathbf{Def}_Z)) \cong CF(Z)\,. \tag{0.0.2}$$

The ring isomorphism in (0.0.1) is given by $[(X, m)] \mapsto (-1)^m\tilde{\chi}(X)$, where $\tilde{\chi}(X) = \chi(X) - 1$ is the reduced Euler characteristic. This will be formalised in Theorem 4.5.6, the proof of which can be found in Section 4.5 of Chapter 4.

The verification of Theorem 4.5.6 uses, in particular, the fact that a compact definable space can be given a CW-structure in a natural way. This is Theorem 1.2.16 of Chapter 1, Section 1.2. We use van den Dries' Good Directions Lemma to show that a definable cell decomposition of a compact definable space can be refined to a *well-based* cell decomposition with certain properties. Then, in turn, we verify that such a well-based definable cell decomposition is a CW-structure with one CW-cell,

$$\sigma_c : [0,1]^{\dim c} \to X,$$

for each definable cell $c$ in the well-based decomposition, and such that $\sigma_c$ is a homeomorphism of the interiors and $\operatorname{im}\sigma_c = \bar{c}$. Our notion of a well-based definable cell decomposition is inspired by the semialgebraic version defined by Schwartz and Sharir in [SS83, p.305]. The result in Theorem 1.2.16 is technically useful for us, but should also be of interest more widely. For example, Berarducci and Fornasiero noted in [BF09] that an obstruction to generalising their theory from the semialgebraic case was the fact that such a result about CW structures in the definable case was not known.

The ring isomorphism $K(\mathbf{SW}(\mathbf{Def}_Z)) \cong CF(Z)$ in (0.0.2) is given by

$$[(X \xrightarrow{p_X} Z, m)] \mapsto (z \mapsto (-1)^m\tilde{\chi}(p_{_X}^{-1}z)),$$

i.e. it is given by the function computing the reduced Euler chracteristic of the fibres of $X$ over $Z$. Theorem 4.7.4 of Chapter 4, Section 4.7, proves the existence of this ring isomorphism. The proof uses the Trivialisation Theorem which tells us that a relative cell decomposition of the morphism $p_{_X} : X \to Z$ exists, and hence that the image is a constructible function. In order to show that 0.0.2 is a well-defined homomorphism, we take advantage of Theorem 4.5.6, together with the fact that the inclusion $\{z\} \to Z$ induces a triangulated functor $\mathbf{SW}(\mathbf{Def}_Z) \to \mathbf{SW}(\mathbf{Def}_z)$ which descends to the Grothendieck groups. This latter fact is a special case of a more general functoriality between relative Spanier–Whitehead categories which is discussed in Chapter 3, Section 3.6. To verify the bijectivity of the homomorphism we use that $CF(Z)$ is generated by indicator functions and then show that the image of a class can be written as a weighted sum of indicator functions over the cells in $Z$ where the weights are given by the reduced Euler characteristic of the fibres over the cells.

In order to construct analogues of the classical Spanier–Whitehead category such as the definable Spanier–Whitehead category $\mathbf{SW}(\mathbf{Def}_*)$ and the relative definable Spanier–Whitehead category $\mathbf{SW}(\mathbf{Def}_Z)$, we develop an axiomatic approach to Spanier–Whitehead categories. In a nutshell the method can be summarised as follows. Given some ambient category $\mathbf{C}$ with some specific properties (existence of suitable finite limits and colimits, existence of an appropriate 'interval object'), we can consider the slice-coslice categories $\mathbf{C}_Z := Z\backslash(\mathbf{C}/Z)$ of $\mathbf{C}$ relative to any fixed $Z \in \mathbf{C}$. These slice-coslice categories have properties analogous to those of the ambient category. Then the verification that for any $Z \in \mathbf{C}$ the Spanier–Whitehead category $\mathbf{SW}(\mathbf{C}_Z)$ exists and is a tensor-triangulated category under smash product follows from the set-up of the ambient category.

This approach provides a neat framework for constructing a plethora of Spanier-Whitehead categories. In Section 3.6 of Chapter 3 we study the base-change functoriality between $\mathbf{SW}(\mathbf{C}_Z)$ and $\mathbf{SW}(\mathbf{C}_{Z'})$ given some morphism $\beta : Z \to Z'$ in a fixed ambient category $\mathbf{C}$. It turns out that there exist triangulated functors:

$$\beta_* : \mathbf{SW}(\mathbf{C}_Z) \to \mathbf{SW}(\mathbf{C}_{Z'})$$

which is a left adjoint to

$$\beta^* : \mathbf{SW}(\mathbf{C}_{Z'}) \to \mathbf{SW}(\mathbf{C}_Z)$$

which is also monoidal. The construction of $\mathbf{SW}(\mathbf{C}_Z)$ is also functorial in the ambient category $\mathbf{C}$ in the following sense. Given two ambient categories $\mathbf{C}$ and $\mathbf{C'}$ and a functor $J : \mathbf{C} \to \mathbf{C'}$ which preserves the interval object and also preserves finite limits and colimits, then $J$ induces, for each $Z \in \mathbf{C}$, a triangulated functor

$$J : \mathbf{SW}(\mathbf{C}_Z) \to \mathbf{SW}(\mathbf{C'}_{J(Z)}).$$

We note that, since objects in $\mathbf{Def}_*$ can be given CW-structures by Theorem 1.2.16, the construction of $\mathbf{SW}(\mathbf{Def}_*)$ could also be done 'manually' by following the standard approach to constructing $\mathbf{SW}(\mathbf{CW}_*)$ described by Margolis in [Mar83]. As we wish to construct various analogues of the classical Spanier–Whithead category, the axiomatic approach is more convenient.

Our axiomatic approach described in Chapter 3 yields Theorem 3.5.8 which tells us that given some ambient category $\mathbf{C}$ with specific properties, the Spanier–Whitehead category $\mathbf{SW}(\mathbf{C}_Z)$ is triangulated for any fixed $Z \in \mathbf{C}$. We have proved this theorem assuming that all finite colimits exist in $\mathbf{C}$. In fact, in order to define a Spanier–Whitehead, only the existence of certain pushouts is used (such as those needed to define suspensions, mapping cones, smash products, etc.). Since we were not able to find a way of elegantly formulating this, we have favoured a set of more general assumptions which produces a neat framework for Spanier–Whitehead constructions. However, in the case $\mathbf{C} = \mathbf{Def}$ for instance, where it is not clear whether or not all finite colimits exist, it is possible to list the specific pushouts which are required in the construction and to verify that they hold in $\mathbf{Def}$.

There exists an alternative approach to an axiomatic Spanier–Whitehead category construction suggested by Dell'Ambrogio in [Del04] with a different set of assumptions. However, this does not seem to be a well-adapted framework for the relative Spanier–Whitehead constructions which we require in order to categorify the constructible functions.

It is the base-change functoriality in the case where $\mathbf{C} = \mathbf{Def}$ which gives rise to triangulated functors

$$\begin{tikzcd} \mathbf{SW}(\mathbf{Def}_Z) \arrow[r, bend left, "\beta_*"] \arrow[r, phantom, "\perp"] & \mathbf{SW}(\mathbf{Def}_{Z'}) \arrow[l, bend left, "\beta^*"] \end{tikzcd}$$

where $\beta_*(\_) := \_ \vee_Z Z'$ and $\beta^*(\_) := \_ \times_{Z'} Z$ are triangulated. In addition, $\beta^*$ is monoidal. The adjoint functors $\beta_*$ and $\beta^*$ induce homomorphisms between the Grothendieck groups which, after identifying these with the constructible functions on $Z$ and on $Z'$ respectively, coincide with the previously discussed operations of the Euler calculus of the same names.

As mentioned earlier, there exists an alternative categorification of $CF(Z)$ by the bounded constructible derived category $D_C(Z;\mathbb{Q})$. We expect that there is a monoidal triangulated comparison functor

$$\mathbf{SW}(\mathbf{Def}_Z) \to D_C(Z;\mathbb{Q})$$

given by

$$(X \xrightarrow{p} Z, n) \mapsto \Sigma^n(Rp_*\mathbb{Q}_X),$$

where $\mathbb{Q}_X$ is the constant sheaf on $X$ with stalk $\mathbb{Q}$, inducing the identity on Grothendieck groups, i.e. on $CF(Z)$. However, we do not believe this to be an equivalence because we expect $\mathbf{SW}(\mathbf{Def}_Z)$ to be a topological triangulated category, whereas $D_C(Z;\mathbb{Q})$ is algebraic. This should follow from a similar approach to that used by Schwede in [Sch10] which explains why the classical Spanier–Whitehead category is not algebraic.

This thesis is divided into four chapters as follows. Chapter 1 provides the basic definitions and properties of o-minimal expansions and definable spaces. We prove that a cell decomposition of a compact definable space can be refined to a CW-structure in Theorem 1.2.16. We also define **Def**, the category of pointed definable spaces, and discuss the existence of definable quotients and definable homotopies in **Def**, noting that these results also hold for the pointed definable category $\mathbf{Def}_*$. The chapter ends with a brief presentation of the constructible functions.

Chapter 2 consists of an overview of the classical Spanier–Whitehead category and its triangulated structure. The axiomatic approach to constructing Spanier–Whitehead categories is presented in Chapter 3. We put forward conditions on an ambient category $\mathbf{C}$ that suffice to construct a Spanier–Whitehead category $\mathbf{SW}(\mathbf{C}_Z)$ for each slice-coslice category $\mathbf{C}_Z := Z\backslash\mathbf{C}/Z$ with $Z \in \mathbf{C}$. We examine $\mathbf{C}_Z$ and coexact (Puppe) mapping sequences in $\mathbf{C}_Z$. We prove that there exist adjoint functors between $\mathbf{C}_Z$ and $\mathbf{C}_{Z'}$ given some morphism between $Z$ and $Z'$ in $\mathbf{C}$. These functors preserve sufficient structure and are given by the pushout and the pullback. We then define $\mathbf{SW}(\mathbf{C}_Z)$ and prove that it is triangulated in Theorem 3.5.8 and tensor-triangulated in Lemma 3.5.10. Finally, in Theorem 3.6.1 we show that the aforementioned adjunction descends to an adjunction between the corresponding Spanier–Whitehead categories where both functors descend to triangulated functors and the pullback functor is monoidal.

Chapter 4 is the culminating chapter of this thesis. It begins with a presentation of a weaker set of assumptions on the ambient category which hold, in particular for **Def**. We also verify that **CW** satisfies these weaker conditions (hence providing an alternative proof that $\mathbf{SW}(\mathbf{CW}_*)$ is a tensor-triangulated category). Once **Def** has been established to be an appropriate ambient category under the weaker assumptions, we consider the absolute case over a point and give the definition of $\mathbf{SW}(\mathbf{Def}_*)$, proving it is a tensor-triangulated category in Theorem 4.4.3. We prove that $K(\mathbf{SW}(\mathbf{Def}_*)) \cong \mathbb{Z}$ in Theorem 4.5.6. We then define the relative Spanier–Whitehead category, and prove that it is tensor triangulated in Theorem 4.6.2. Finally, the crucial result that $K(\mathbf{SW}(\mathbf{Def}_Z)) \cong CF(Z)$ is proved in Theorem 4.7.4. We conclude by discussing the lifting of the Euler calculus to the relative definable Spanier–Whitehead categories.

# Chapter 1

# Definable spaces

The notion of an o-minimal structure, developed by Pillay and Steinhorn during the 1980s (in for example [PS84]), is a generalisation of the geometries of classes such as the semialgebraic and semianalytic sets. These classes of sets have many nice properties. The semialgebraic subsets of $\mathbb{R}^n$ are the subsets defined by finite Boolean combinations of subsets cut out by real polynomial equations and inequalities. This class of sets is stable under Boolean operators and projections. The fact that it is closed under projections is known as the Tarski-Seidenberg Projection Property and distinguishes the semialgebraic sets from the algebraic sets. The semialgebraic sets also have an important finiteness property in that each set has a finite number of connected components and each of these is also semialgebraic. In addition to having pleasing properties, the topology of the semialgebraic sets is 'straightforward' since the definition ensures no pathological phenomena arise. A classic example of the type of wayward behaviour that does not exist in the class of semialgebraic sets is given by the graph of the function $x \mapsto \sin(1/x)$ for $x > 0$. Despite the graph being homeomorphic to the interval, there is a wildness to it as demonstrated by the fact that its closure in $\mathbb{R}^2$ is not homeomorphic to the closed interval. Following the model of the class of semialgebraic sets, o-minimal structures rule out disagreeable behaviour such as this.

Roughly an o-minimal structure on $\mathbb{R}$ is a collection of well-behaved subsets of the Euclidean space with which one can perform standard geometric and topological constructions. An o-minimal structure on $\mathbb{R}$ which contains the graphs of addition and multiplication is called an o-minimal expansion of $\mathbb{R}$. This latter notion was first established by van den Dries in [vdD84] as a framework for developing Grothendieck's famous *topologie modérée* as in his "Esquisse d'un Programme" of 1984, [Gro97]. Thus the o-minimal structures allow one to develop a *tame topology* in which undesirable behaviour is eliminated.

In the last three decades, since Wilkie proved in 1991 in [Wil96] that the real exponential field is o-minimal, many new and remarkable o-minimal structures have been proven to exist and the theory of o-minimal structures has truly blossomed.

There is an extensive survey of the subject by van den Dries in [vdD98] which we refer to for the elementary definitions and properties. O-minimal structures have many important properties such as the existence of nice cell decompositions (where the closure of any cell is a union of cells) and a well-defined Euler characteristic which is independent of the chosen cell decomposition. Note that the notion of o-minimal cell decomposition is related to, but not the same as, the usual notion of cell decomposition in topology.

Throughout we will work in some fixed o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$. We will say that a space is *definable* if it belongs to this fixed o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$ and give it the subspace topology of the usual Euclidean topology on $\mathbb{R}^m$. A map is definable if its graph is. Definable spaces have many other nice properties, such as the Trivialisation Theorem which says that one can find a cell decomposition of the codomain of a continuous definable map which is compatible with the decomposition of the domain. Of course there exists a notion of o-minimal homotopy (a homotopy definable in some o-minimal expansion of $\mathbb{R}$), and there are useful relationships between semialgebraic, o-minimal and classical homotopies as discussed by Otero and Baro in [BO10]. There also exists a definable Triangulation Theorem for o-minimal expansions of $\mathbb{R}$. Coste provides a neat overview of o-minimal geometry in [Cos00]. For an in-depth study of the theory of o-minimal structures and tame topology, we refer the reader to [vdD98].

We prove that any compact space which is definable has an associated CW structure in Theorem 1.2.16. This is done by showing that a definable cell decomposition of a compact definable space can be refined to be *well-based*, and that such well-based decompositions have an associated CW structure with one CW-cell for each definable cell. This idea of a well-based definable cell decomposition is motivated by the notion of a well-based semialgebraic cell decomposition defined by Schwartz and Sharir in [SS83, p.305]. In order to prove that any definable cell decomposition can be refined to a well-based one, we use the Good Directions Lemma, [vdD98, p.117], and to prove it has an associated CW-structure, we use an extension property for cells. The result in Theorem 1.2.16 will be used, in particular, to prove that the Grothendieck group of the definable version of the Spanier–Whitehead category is generated by the class of the 0-sphere.

In our approach to constructing the definable Spanier–Whitehead category, objects are sourced from the definable category **Def** of compact definable spaces. This definable category has nice properties such as finite limits and pushouts along closed inclusions, so constructions such as the definable suspension of an object and the definable mapping cone on a morphism can be considered in **Def**. There is a notion of homotopy in **Def** and if two maps are definably homotopic, then they are also homotopic in the classical sense. Naturally, we also define the pointed definable category $\mathbf{Def}_*$ and since the aforementioned properties hold for $\mathbf{Def}_*$ too, we can consider Puppe sequences in $\mathbf{Def}_*$. These Puppe sequences (or mapping cone sequences) in $\mathbf{Def}_*$ will play a significant role in the construction of the definable Spanier–Whitehead category $\mathbf{SW}(\mathbf{Def}_*)$ as will be seen in Chapters 3 and 4.

Our main aim, to provide a homotopical categorification of the Euler calculus, will be achieved by lifting the operations of the Euler calculus to functors between relative versions of the definable Spanier–Whitehead category. The Euler calculus can be thought of as a theory of integration for groups of constructible functions on compact definable spaces, and the Euler integral of a constructible function is the pushforward to a point. To lift the operations of the Euler calculus, we will use that the definable Spanier–Whitehead category relative to a fixed definable space $Z$ has Grothendieck group isomorphic to the ring of constructible functions on $Z$. This crucial result will be proved in Section 4.7 of Chapter 4.

This chapter provides the necessary background in o-minimal structures together with the key results about definable spaces on which our approach rests. It is structured as follows. Section 1.1, taking van den Dries' presentation of the subject in [vdD98] as a model, lays out the basic definitions and properties of o-minimal expansions of $\mathbb{R}$. In Section 1.2 we prove that any definable cell decomposition can be refined to a well-based decomposition, and hence that any compact definable space can be given a CW structure. The definition and properties of the definable category $\mathbf{Def}$ of compact definable spaces are set out in Section 1.3. In particular we discuss the existence of definable quotients in $\mathbf{Def}$. In this section we also examine the underlying topology of definable spaces and the notion of a definable homotopy. Our treatment of definable homotopies largely follows that of Baro and Otero in [BO10]. In addition we give the definition of the pointed definable category $\mathbf{Def}_*$ and look at Puppe sequences in $\mathbf{Def}_*$. Finally, Section 1.4 contains a brief discussion about the constructible functions $CF(X)$ on a compact definable space and the fact that they form a ring under point-wise addition and multiplication of functions.

## 1.1 Definitions and elementary properties

A **boolean algebra of subsets of a set** $X$ is a nonempty collection $\mathcal{C}$ of subsets of $X$ such that if $A, B \in \mathcal{C}$, then $A \cup B \in \mathcal{C}$ and $X - A \in \mathcal{C}$. It follows immediately that $\emptyset \in \mathcal{C}$ and $X \in \mathcal{C}$, and that if $A, B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$.

**1.1.1 Definition.** A **structure** on $\mathbb{R}$ is defined to be a sequence $\mathcal{S} = (\mathcal{S}_m)_{m \in \mathbb{N}}$

such that for each $m \geq 0$, the following axioms hold:

**S.1** $\mathcal{S}_m$ is a boolean algebra of subsets of $\mathbb{R}^m$;

**S.2** If $A \in \mathcal{S}_m$, then $\mathbb{R} \times A$ and $A \times \mathbb{R}$ belong to $\mathcal{S}_{m+1}$;

**S.3** $\{(x_1, ..., x_m) \in \mathbb{R}^m : x_1 = x_m\} \in \mathcal{S}_m$;

**S.4** If $A \in \mathcal{S}_{m+1}$, then $\pi(A) \in \mathcal{S}_m$, where $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$ is the projection map onto the first $m$ coordinates.

We say that $A \subseteq \mathbb{R}^m$ belongs to $\mathcal{S}$ if it belongs to $\mathcal{S}_m$. A map $f : A \to B$, with $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$, belongs to $\mathcal{S}$ if its graph $\Gamma(f) = \{(x, y) : y = f(x)\} \subseteq \mathbb{R}^{m+n}$ belongs to $\mathcal{S}_{m+n}$.

The following lemmas provide us with some elementary facts about these general structures on $\mathbb{R}$.

**1.1.2 Lemma** ([vdD98, p.13])**.**

*(i)* *If* $A \in \mathcal{S}_m$ *and* $B \in \mathcal{S}_n$, *then* $A \times B \in \mathcal{S}_{m+n}$.

*(ii)* *For* $1 \leq i < j \leq m$, *the diagonal* $\Delta_{ij} := \{(x_1, ..., x_m) \in \mathbb{R}^m : x_i = x_j\}$ *belongs to* $\mathcal{S}$.

*(iii)* *("Permuting and identifying variables are allowed".) Let* $B \in \mathcal{S}_n$, *and let* $i(1), ..., i(n) \in \{1, ..., m\}$. *If* $A \subseteq \mathbb{R}^m$ *is the set defined by the condition*

$$(x_1, ..., x_m) \in A \Leftrightarrow \left(x_{i(1)}, ..., x_{i(n)}\right) \in B,$$

*then* $A$ *belongs to* $\mathcal{S}$.

**1.1.3 Lemma** ([vdD98, p.14])**.** *Let* $S \subseteq \mathbb{R}^m$ *and let* $f : S \to \mathbb{R}^n$ *be a map that belongs to* $\mathcal{S}$ *(i.e.* $\Gamma(f) \in \mathcal{S}_{m+n}$*). Then the following properties hold:*

*(i)* $S \in \mathcal{S}_m$;

*(ii)* *If* $A \subseteq S$ *and* $A \in \mathcal{S}_m$, *then* $f(A) \in \mathcal{S}_n$ *and the restriction of the map,* $f|_A$, *belongs to* $\mathcal{S}$;

*(iii)* *If* $B \in \mathcal{S}_n$, *then* $f^{-1}(B) \in \mathcal{S}_m$;

*(iv)* *If* $f$ *is injective then its inverse* $f^{-1}$ *belongs to* $\mathcal{S}$;

*(v)* *If we have* $f(S) \subseteq T \subseteq \mathbb{R}^n$ *and another map* $g : T \to \mathbb{R}^p$ *which belongs to* $\mathcal{S}$, *then the composition* $g \circ f : S \to \mathbb{R}^p$ *also belongs to* $\mathcal{S}$.

**1.1.4 Definition.** An **o-minimal structure** on $\mathbb{R}$ is by definition a structure $\mathcal{S}$ on $\mathbb{R}$ such that

**O.1** $\{(x, y) \in \mathbb{R}^2 : x < y\} \in \mathcal{S}_2$,

**O.2** the sets in $\mathcal{S}_1$ are exactly the finite unions of open intervals and points.

**1.1.5 Example.**

1. The simplest o-minimal structure on $\mathbb{R}$ is the class of **simple sets** defined just using constants. This is an o-minimal structure which contains very few sets, consisting of just the dense linearly ordered nonempty sets without endpoints. A simple function $f : \mathbb{R}^n \to \mathbb{R}$ is either a coordinate projection

$$(x_1, \ldots, x_n) \mapsto x_i,$$

or a constant. A basic simple set $A \subset \mathbb{R}^n$ is a subset cut out by a finite (possibly zero) number of equalities and inequalities of the form $f(x) = g(x)$ or $f(x) < g(x)$ where $f$ and $g$ are simple functions. A simple set is any finite union of basic simple sets. It is easy to check that the simple sets are an o-minimal structure on $\mathbb{R}$, but that they neither contain the graph $x + y = z$ of addition nor the graph $xy = z$ of multiplication. See [vdD98, Ch.1, §6] for further details.

2. The semilinear sets are an example of an o-minimal structure on $\mathbb{R}$ which includes the graphs of $0 : \mathbb{R}^0 \to \mathbb{R}$, $- : \mathbb{R} \to \mathbb{R}$, and $+ : \mathbb{R}^2 \to \mathbb{R}$. They are defined in exactly the same way as the simple sets but where we allow $f$ and $g$ to be affine linear functions $\mathbb{R}^n \to \mathbb{R}$, i.e. functions of the form

$$(x_1, \ldots, x_n) \mapsto a_1x_1 + \cdots + a_nx_n + b.$$

(In this case we can reduce to considering equalities and inequalities of the form $f(x) = 0$ or $f(x) > 0$ when defining basic semilinear sets, because the difference of affine linear functions is affine linear.) Again, it is not difficult to verify that the semilinear sets form an o-minimal structure. The graph $x + y = z$ of addition is a semilinear set, but the graph $xy = z$ of multiplication is not. So this o-minimal structure expands $\mathbb{R}$ considered as an ordered abelian group, but not as an ordered ring. When we refer to an 'o-minimal expansion of $\mathbb{R}$', we will always mean an o-minimal structure expanding $\mathbb{R}$ as an ordered ring, i.e. an o-minimal structure containing the graphs of addition and of multiplication. See [vdD98, Ch.1, §7] for further details.

**1.1.6 Definition.** If an o-minimal structure $\mathcal{R}$ on $\mathbb{R}$ includes the graphs of $0 : \mathbb{R}^0 \to \mathbb{R}$, $- : \mathbb{R} \to \mathbb{R}$, $+ : \mathbb{R}^2 \to \mathbb{R}$, and the graph of multiplication $\times : \mathbb{R}^2 \to \mathbb{R}$, then we say that $\mathcal{R}$ is **an o-minimal expansion of the real field** $\mathbb{R}$.

The smallest o-minimal expansion of $\mathbb{R}$ is the class $\mathcal{R}^{\mathcal{SA}}$ of *semialgebraic sets* (in the sense that it is contained within all other expansions).

**1.1.7 Definition. The semialgebraic sets** are the subsets of $\mathbb{R}^n$ which are finite Boolean combinations of subsets cut out by (real) polynomial equations or inequalities, i.e. subsets of the form

$$\bigcup_{i \in I} \bigcap_{j \in J_i} X_{ij},$$

where $I$ and $J_i$ are finite sets, and each $X_{ij}$ is either of the form $\{p(x_1, ..., x_n) = 0\}$ or the form $\{p(x_1, ..., x_n) > 0\}$ for some polynomial $p \in \mathbb{R}[x_1, ..., x_n]$.

For the semialgebraic sets, the axioms (S1)-(S3) of Definition 1.1.1 and (O1)-(O2) of Definition 1.1.4 are easy to verify. Axiom (S4) of Definition 1.1.1 is essentially the Tarski-Seidenberg Theorem [vdD98, p.37]. This is a key property which distinguishes semialgebraic from algebraic sets.

Below are some examples of semialgebraic spaces.

**1.1.8 Example.** By definition the semialgebraic subsets of $\mathbb{R}$ are the unions of finitely many points and open intervals.

**1.1.9 Example.** Any algebraic subset of $\mathbb{R}^n$ (defined by polynomial equations) is semialgebraic.

**1.1.10 Example.** The interior of a standard simplex in $\mathbb{R}^n$ is semialgebraic.

**1.1.11 Example.** The following are not semialgebraic:

$$\{(x, y) \in \mathbb{R}^2 : y = \sin x\}$$

and

$$\bigcup_{n \in \mathbb{N}} X_n \text{ where } X_n = \{(x, y) \in \mathbb{R}^2 : y = nx\}.$$

We will say that $X \subseteq \mathbb{R}^m$ is a **definable space** if it belongs to an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$, i.e. if $X \in \mathcal{R}_m$. We give $X$ the subspace topology of the usual Euclidean topology on $\mathbb{R}^m$. Similarly, we say that a map $f : X \to Y$, where $X \in \mathcal{R}_m$ and $Y \in \mathcal{R}_n$, is a **definable map** if its graph,

$$\varGamma(f) = \{(x, y) : y = f(x)\},$$

is definable, i.e. if $\Gamma(f) \in \mathcal{R}_{m+n}$. We note that a definable map need not be continuous with respect to the subspace topologies on the source and target. However, we will almost always be interested in continuous definable maps.

From Lemma 1.1.2 and Lemma 1.1.3 we see that definable spaces and definable maps have many nice properties. Other useful facts are:

1. if $X$ is definable then the closure $\overline{X}$ and the interior $X^0$ are definable, and as a result, so are the boundary $\partial X = \overline{X} - X^0$ and the frontier fr$X = \overline{X} - X$;

2. if $f, g : X \to \mathbb{R}$ are two definable functions, then $kf$ for any $k \in \mathbb{R}$, $f + g$, $f \times g$, $\max\{f, g\}$, and $\min\{f, g\}$ are also definable.

In general, an o-minimal expansion $\mathcal{R}$ is defined either with parameters from the underlying real field or without parameters (i.e. without constants). In particular, we have:

**1.1.12 Lemma** ([vdD98, p.37]). *The sets definable with parameters from* $\mathbb{R}$ *are just the semialgebraic sets.*

To avoid going into too much detail, an example is probably the easiest way to get a grasp of this notion. The paraboloid $x^2 + y^2 = z$ (which can be defined over any field, i.e. without constants) is an example of a space definable without parameters. If we allow parameters (from $\mathbb{R}$), then we also consider the fibres of all the standard projections (i.e. we also allow polynomial equations such as $x^2 + y^2 = 1$, $x^2 + y^2 = 2$), and we obtain a parametrised space which can be thought of as a family of fibres. Thus, by definition, the semialgebraic sets are the sets which are definable with parameters from $\mathbb{R}$ (i.e. the sets in $\mathbb{R}^n$ cut out by polynomials with coefficients in $\mathbb{R}$). For the rest of the paper we fix definable to mean **definable in an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$ with parameters**, and the expansion concerned will always be clear from the context.

We now present some basic properties of definable spaces.

**1.1.13 Lemma.** *Suppose $X$ is a compact definable space and $f : X \to Y$ is a continuous definable bijection. Then $f$ is a definable homeomorphism, i.e. it has a continuous definable inverse.*

*Proof.* We have assumed that $f$ is a bijection and that $X$ is compact, and we know that a definable space is Hausdorff because it is a subspace of some Euclidean space. So in particular $Y$ is Hausdorff. Thus $f$ is a homeomorphism, i.e. $f^{-1}$ is continuous. Clearly $f^{-1}$ is definable since its graph is the reflection of the graph of $f$ in $X \times Y$. $\square$

**1.1.14 Remark.** We recall that we consider *continuous* definable maps. As mentioned previously, in general a definable map is not necessarily a continuous function. So a general definable homeomorphism is just a definable bijection between definable spaces.

One of the most important characteristics of definable spaces is the existence of definable cell decompositions.

**1.1.15 Definition.** The **definable cells** in $\mathbb{R}^n$ are defined inductively on $n$:

1. $\{0\}$ is a cell in $\mathbb{R}^0$;

2. the cells in $\mathbb{R}^1$ are just the points $\{r\}$, and the open intervals $(a, b)$;

3. if $C \subset \mathbb{R}^n$ is a cell, then

   (a) $C \times \mathbb{R}$ is a cell;

   (b) if $f : C \to \mathbb{R}$ is a continuous definable map, then the sets of those $(x, t) \in C \times \mathbb{R}$ such that $f(x) = t$, such that $f(x) < t$, and such that $f(x) > t$ are cells;

   (c) if $f, g : C \to \mathbb{R}$ are continuous definable functions with $f(x) < g(x)$ for all $x \in C$ then the set of $(x, t) \in C \times \mathbb{R}$ such that $f(x) < t < g(x)$ is a cell.

With this construction, each cell is definable.

**1.1.16 Example.** The open disk $\{x \in \mathbb{R}^n : ||x|| < 1\}$ is a definable cell in $\mathbb{R}^n$, as is the interior of a standard simplex in $\mathbb{R}^n$.

A definable space $X$ is said to be **definably connected** if $X$ is definable and $X$ is not a union of two disjoint nonempty definable open subsets of $X$.

**1.1.17 Lemma** ([vdD98, p.51])**.** *A definable cell is definably connected.*

**1.1.18 Definition.** A **definable cell decomposition** of $\mathbb{R}^n$ is defined inductively on $n$:

1. A cell decomposition of $\mathbb{R}$ is a finite partition of $\mathbb{R}$ into cells (points and open intervals);

2. A cell decomposition of $\mathbb{R}^{n+1}$ is a finite partition $\{C_i : i \in I\}$ of $\mathbb{R}^{n+1}$ into cells such that $\{\pi(C_i) : i \in I\}$ is a cell decomposition of $\mathbb{R}^n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first $n$ coordinates.

We say that a cell decomposition of $\mathbb{R}^n$ is **compatible** with $A \subset \mathbb{R}^n$ if $A$ is a union of cells. We will often just say 'a cell decomposition of $A$'. We have the following theorem on the existence of definable cell decompositions.

**1.1.19 Theorem** (Cell Decomposition Theorem, [vdD98, p.52])**.** *Given definable $A_1, ..., A_n \subset \mathbb{R}^n$ there is a cell decomposition of $\mathbb{R}^n$ compatible with each of the $A_i$. Moreover, the cell decomposition can be chosen to satisfy the frontier condition, i.e. so that the closure of any cell is a union of cells.*

The following theorem is a key result which tells us that, given a definable map, there exist cell decompositions of the domain and codomain which are compatible.

**1.1.20 Theorem** (Trivialisation Theorem, [vdD98, p.147])**.** *Suppose $f : A \to B$ is a continuous definable map between definable spaces. Then $B$ has a cell decomposition $B = B_1 \sqcup ... \sqcup B_k$ such that there exist definable sets $F_i$ and definable homeomorphisms $h_i : f^{-1}B_i \to B_i \times F_i$ making the diagram*

$$\begin{tikzcd} f^{-1}B_i \arrow[rr, "h_i"] \arrow[dr, "f"'] & & B_i \times F_i \arrow[dl, "p_1"] \\ & B_i & \end{tikzcd}$$

*commute (where $p_1$ is the projection onto the first factor).*

Here each cell in $B_i \times F_i$ maps by projection to a cell in $B_i$. So, given a continuous definable map from $A$ to $B$, there is a 'nice' cell decomposition in B such that the preimage of each of the cells is just the product of that cell with some fibre, i.e. there is a cell decomposition such that each cell is simply a projection. Thus the trivialisation theoreom can be thought of as a relative version of the cell decomposition theorem.

Another important fact about definable spaces is that they have well-defined (compactly supported) Euler characteristic. It is given by the alternating sum of numbers of cells of each dimension in a cell decomposition. Moreover, this definition is independent of the choice of the cell decomposition.

**1.1.21 Proposition.** *Let $X$ be a definable space. Then the compactly supported Euler characteristic*

$$\chi_c(X) = \sum_{i \in I} (-1)^{\dim C_i}$$

*is well-defined and is independent of the cell decomposition $X = \bigsqcup_{i \in I} C_i$ used to compute it.*

*Proof.* This proposition is a reformulation of Proposition 2.2 and Lemma 2.6 in [vdD98]. We refer the reader to p.70-71 of [vdD98] for the proofs. □

**1.1.22 Example.** Considering the trivial cell decomposition of $\mathbb{R}^n$ (with only one $n$-dimensional cell), we have that

$$\chi_c(\mathbb{R}^n) = (-1)^n.$$

So, unlike the usual Euler characteristic, the compactly supported Euler characteristic is not a homotopy invariant. Alternatively we can take, as cell decomposition of $\mathbb{R}^n$, a subdivision into two $n$-dimensional cells and one $(n-1)$-dimensional cell. Then we have

$$\chi_c(\mathbb{R}^n) = (-1)^n + (-1)^{n-1} + (-1)^n = (-1)^n$$

as expected since $\chi_c$ does not depend on the choice of cell decomposition.

**1.1.23 Example.** The $n$-sphere $S^d = \{x_0^2 + ... + x_d^2 = 1\}$ is a definable subset of $\mathbb{R}^{d+1}$. We observe that the image under projection away from the last factor is the union of an open $d$-cell and $S^{d-1}$. It follows that we can inductively construct a cell decomposition of $S^d$ with two $i$-cells for each $0 \leq i \leq d$. Hence

$$\chi_c(S^d) = \begin{cases} 2 & d \text{ even} \\ 0 & d \text{ odd.} \end{cases}$$

## 1.2 The CW-structure of a compact definable space

In this section we prove that a compact definable space has a CW structure. This will be used in the proof of Theorem 4.5.6 which is a key result.

From now on, we assume that $X$ is a compact definable subspace of $\mathbb{R}^n$ with some given cell decomposition as defined in Definition 1.1.18 (i.e. a cell decomposition of $\mathbb{R}^n$ with $X$ as a union of cells). Since $X$ is compact, if $c$ is a cell in $X$, then its closure $\bar{c}$ is also in $X$. We want to understand when $X$ can be given an associated CW-structure such that each definable $k$-cell in $X$ is the interior of a unique CW $k$-cell. More precisely we want to determine when there exists definable homeomorphism

$$(0,1)^k \xrightarrow{\sigma} c$$

which extends (necessarily uniquely) to a continuous definable map

$$[0,1]^k \xrightarrow{\bar{\sigma}} \bar{c} \subset X.$$

### 1.2.1 Well-based definable cell decompositions

**1.2.2 Definition.** Assume $X \subset \mathbb{R}^n$ is a compact definable subspace with cell decomposition $K$. A **graph-cell** in $K$ is a cell of the form

$$c = \{(x, f(x)) \in \mathbb{R}^m : x \in \pi c\}$$

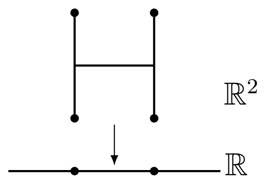
where $f : \pi c \to \mathbb{R}$ (so $c$ is of the same dimension as $\pi c$). A **finite band-cell** in $K$ is a cell of the form

$$c = \{(x, y) \in \mathbb{R}^m : x \in \pi c \textit{ and } f_1(x) < y < f_2(x)\}$$

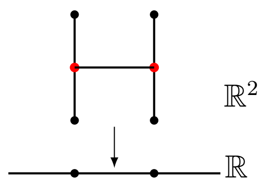
where $f_1, f_2 : \pi c \to \mathbb{R}$ with $f_1 < f_2$.

In order to be able to associate a CW-structure to a definable cell decomposition, the first basic property required is that the boundary of each cell should be a union of lower dimensional cells. The following example shows that this is not necessarily the case for definable cell decompositions.

**1.2.3 Example.** Consider the following definable cell decomposition of $\mathbb{R}^2$




Clearly the boundary of the horizontal cell is not a union of lower dimensional cells. However, we can refine the above cell decomposition as follows




This refinement is definable and such that the boundary of each cell consists of a union of lower dimensional cells.

**1.2.4 Remark.** We note that, given some finite set of definable subspaces (of some $\mathbb{R}^n$), it is always possible to choose a cell decomposition such that each of the subspaces is a union of cells. Hence, given some cell decomposition, we can always take a refinement of that decomposition.

In fact, it is always possible to refine a given definable cell decomposition as in Example 1.2.3.

**1.2.5 Lemma.** *Any cell decomposition $K$ can be refined to a cell decomposition $K'$ such that for each cell $c$ in $K$, its closure $\bar{c}$ is a union of cells*

*Proof.* Firstly we note that, for any cell $c$ in a definable cell decomposition $K$ of compact definable $X \subset \mathbb{R}^n$, the closure $\bar{c}$ is definable.

Suppose $d = \dim X$, and let

$$X_1 = X - \bigcup_{\dim c = d} c,$$

i.e. remove the top dimensional cells. Also let $A_i \subset X_i$ be the boundary $\partial c_i$ for each cell $c_i$ with $\dim c_i = d$.

Then, using the Cell Decomposition Theorem 1.1.19, we can refine the decomposition of $X_1$ so that each $A_i$ is a union of cells. The top dimensional cells in the decomposition remain unchanged, but their boundaries are now unions of cells.

We then repeat this for one dimension lower with

$$X_2 = X - \bigcup_{\dim c \geq d-1} c,$$

and taking $A_i \subset X_2$ to be the boundary $\partial c_i$ for each cell $c_i$ with $\dim c_i = d - 1$.

Thus the boundaries of cells of dimension $d - 1$ are also unions of cells (and since at each stage we are refining the decomposition, the boundaries of cells of dimension $d$ remain unions of cells).

The result then follows by induction.

□

**1.2.6 Definition.** A cell decomposition with the closure property, i.e. the closure each cell in the decomposition is a union of cells, will be called a **CDCP**.

From now on we assume that each cell decomposition is a CDCP.

**1.2.7 Definition.** Assume $X \subset \mathbb{R}^n$ is a compact definable subspace.

(i) A definable cell decomposition of $X$ is said to have an **associated CW-structure** (with finitely many cells and definable attaching maps) if it has the cell extension property, i.e. if for each definable cell $c$ there is a choice

$$\sigma : (0,1)^{\dim c} \to c$$

of definable homeomorphism for each cell $c$ in $X$ which extends to a continuous definable (attaching) map

$$\bar{\sigma} : [0,1]^{\dim c} \to \bar{c}.$$

(ii) A definable cell decomposition of $X$ is said to be **well-based** if for each graph cell

$$c = \{(x, f(x)) : x \in \pi c\} \subset \pi^k X \subset \mathbb{R}^{n-k},$$

where $\pi$ is the orthogonal projection along the final coordinate direction, the continuous definable function $f : \pi c \to \mathbb{R}$ has a continuous definable extension $\bar{f} : \bar{\pi c} \to \mathbb{R}$.

**1.2.8 Remark.** Our notion of a well-based definable cell decomposition is inspired by that of a *well-based semialgebraic cell decomposition* as defined in [SS83, p.305].

**1.2.9 Lemma.** *Given a compact definable subspace $X \subset \mathbb{R}^n$, we have*

$$(ii) \Rightarrow (i).$$

*Proof.* We prove this by induction over the dimensions of cells. For $n = 1$, the case is clear. Assume the statement holds for $n-1$, and consider a $k$-cell $c$ in $X$. By induction, for $\pi c$ we have an attaching map $\bar{\sigma}_{\pi c}$. If $c$ is a graph cell then by assumption we have a continuous definable map $\bar{f} : \bar{\pi c} \to \mathbb{R}$, and we define

$$\bar{\sigma}_c(t_1, ..., t_k) = (\bar{\sigma_{\pi c}}(t_1, ..., t_k), \bar{f}(t_1, ..., t_k)) \in \mathbb{R}^n$$

for $(t_1, ..., t_k) \in [0,1]^k$. If $c$ is a band cell, say

$$c = \{(x, t) : x \in \pi c \text{ and } f(x) < t < g(x)\}$$

for continuous definable $f, g : \pi c \to \mathbb{R}$, then we define

$$\bar{\sigma}_c(t_1, ..., t_k) = (\bar{\sigma_{\pi c}}(t_1, ..., t_{k-1}), (1-t_k)\bar{f}(t_1, ..., t_{k-1}) + t_k \bar{g}(t_1, ..., t_{k-1}))$$

where $\bar{f}, \bar{g} : [0,1]^{k-1} \to \mathbb{R}$ are the continuous definable extensions giving the attaching maps for the lower and upper bounding graph cells.

□

**1.2.10 The CW-structure of a well-based definable cell decomposition** By Lemma 1.2.9, in order to prove that a definable cell decomposition of a compact definable space has an associated CW-structure, we need to show that the decomposition can be refined to a well-based cell decomposition. The following example shows that a change of coordinates is required in order to obtain a cell decomposition which is well-based.

**1.2.11 Example.** Consider

$$X = \{((x,y), \frac{y}{\sqrt{x^2+y^2}}) : 0 < x, y < 1\} \subset \mathbb{R}^3$$

and take the closure $\bar{X}$. Choose a definable cell decomposition of $\mathbb{R}^3$ with $X$ as a cell and $\bar{X}$ as a union of cells. Although $X$ is a graph cell, the intersection of $\bar{X}$ with the fibre over the point $(0,0)$ contains a band-cell. Therefore this decomposition of $X \subset \mathbb{R}^3$ is not well-based. To obtain a well-based decomposition, a change in coordinates is required.

**1.2.12 Theorem** (Good Directions Lemma, [vdD98, p.117])**.** *Let $A \subseteq \mathbb{R}^{n+1}$ be some definable space with* $\dim(A) < n+1$.

*Then, for each $x \in \mathbb{R}^n$ with $|x| < 1$, define $v(x)$ to be the point on the unit sphere in $\mathbb{R}^{n+1}$ which lies directly above $x$, i.e. $v(x) := (x_1, ..., x_n, \sqrt{1-|x|^2})$, so that $v(x) \in \mathbb{R}^{n+1}$ and $|v(x)| = 1$.*

*Also define $B \subseteq \mathbb{R}^n$ to be a box contained in the disc $|x| < 1$, and $p \in \mathbb{R}^{n+1}$ to be some point in $\mathbb{R}^{n+1}$.*

*There exists $x \in B$ such that, for each point $p \in \mathbb{R}^{n+1}$, the set $\{t \in \mathbb{R} : p + t \cdot v(x) \in A\}$ is finite.*

**1.2.13 Definition. A good direction for** $A$ is the direction $v(x)$ with $x \in B$ such that $\{t \in \mathbb{R} : p + t \cdot v(x) \in A\}$ is finite for each $p \in \mathbb{R}^{n+1}$.

Thus every line in $\mathbb{R}^{n+1}$ in a good direction for $A$ will intersect $A$ in only finitely many points

The following is a technical result which we will require.

**1.2.14 Lemma.** *Let $x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}$, and for sufficiently small $\epsilon$ let $B_\epsilon$ be a $\epsilon$-cube (with diameter $2\epsilon$ where $\epsilon > 0$), centered about $x$.*

*If $x \in \partial c$, then the $\epsilon$-cube $B_\epsilon(x)$ centered about $x$ is connected, in particular definably connected, for* $0 < \epsilon \leq 1$.

*Proof.* We prove that, for any $x \in \overline{c}$,

$$B_\epsilon \cap c$$

is a cell (for sufficiently small $\epsilon$), and hence is connected. We proceed inductively and assume that $\pi(B_\epsilon) \cap \pi c$ is a cell. Then $x' = (x'_1, ..., x'_{n+1}) \in B_\epsilon \cap c$ if and only if $(x'_1, ..., x'_n) \in \pi(B_\epsilon) \cap \pi(c)$ and either

(i) $x^{,}_{n+1} = f(x^{,}_1, ..., x^{,}_n)$, or

(ii) $\max\{f(x^{,}_1, ..., x^{,}_n), x_{n+1} - \epsilon\} \leq x^{,}_{n+1} \leq \min\{g(x^{,}_1, ..., x^{,}_n), x_{n+1} + \epsilon\}$.

We know that $f, g, x_{n+1} \pm \epsilon$ and $\max, \min$ are all definable. So we have two cases; either we have a graph-cell or a band-cell (cases $(i)$ and $(ii)$ respectively). By Lemma 1.1.17, we know that definable cells are definably connected. Thus $B_\epsilon \cap c$ is connected, as required. □

**1.2.15 Proposition.** *Assume $X \subset \mathbb{R}^n$ is a compact definable subspace with a given cell decomposition. There exists a set of orthogonal projections*

$$\mathbb{R}^n \to \mathbb{R}^{n-1} \to ... \to \mathbb{R}$$

*and a well-based CDCP with respect to these new coordinates which refines the original cell decomposition, i.e. the original cells are unions of cells in the new decomposition.*

*Proof.* We prove this proposition by induction on the dimension $n$. When $n = 0$, the case is clear since the only definable space is a point. Let $A \subset X$ be the union of cells of dimension less than the top dimension i.e. less than $\dim X - 1$, then choose a good direction for $A$, and let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the orthogonal projection in this good direction.

Choose a cell decomposition of $X$ starting with the projection $\pi$ in the good direction and refining the original cell decomposition

By induction, we can refine the cell decomposition of $\pi X \subset \mathbb{R}^{n-1}$ (changing coordinates if necessary) to one which is well-based.

Consider the induced refinement of the cell decomposition of $X$ over this well-based refinement of the decomposition of $\pi X$ (where the cells have the form $c'' \cap \pi^{-1}c'$, where $c''$ is an original cell in $X$ and $c'$ is a refined cell in $\pi X$).

We want to show that this refined cell decomposition of $X$ is well-based. For the graph-cells in $\pi X$ and its images, this is immediate (since we chose a good direction). Hence we only need to check that this is true for the graph-cells in $X$, i.e. for cells of the form

$$c = \{(x, f(x)) : x \in \pi c\}$$

for a definable map $f : \pi c \to \mathbb{R}$.

We recall that we are assuming that all cell decompositions are CDCPs. Consider the closure $\bar{c}$ in $X$. This is a union of cells by Lemma 1.2.5. Thus we have $\partial c = \bar{c} - c \subset A$.

Since $\pi$ is a good direction,

$$|\pi^{-1}y \cap \partial c| < \infty$$

for any $y \in \pi X$, in particular for $y \in \partial(\pi c)$. This is a discrete set, and because $c$ is a graph-cell, we have that $\pi^{-1}x \cap c = \{(x, f(x))\}$ is a single point for each $x \in \pi c$. Hence there exists a unique point

$$(y, \bar{f}(y)) \in \pi^{-1}y \cap \partial c$$

such that $B_\epsilon((y, \bar{f}(y))) \cap c \neq \emptyset$ for all $\epsilon > 0$. This is because for sufficiently small $\epsilon > 0$, the union

$$\bigcup_{z \in \pi^1 y \cap A} B_\epsilon(x) \subset \pi^{-1} B_\epsilon(y)$$

is disconnected, but

$$c \cap \pi^{-1} B_\epsilon(y) \underset{\simeq}{\xrightarrow{\pi}} \pi c \cap B_\epsilon(y)$$

is connected (by Lemma 1.2.14).

This defines a function $\bar{f} : \bar{\pi c} \to \mathbb{R}$ such that $\bar{f}(y) = f(y)$ for $y \in \pi c$ and by above for $y \in \partial(\pi c)$ and

$$\{(y, \bar{f}(y)) : y \in \bar{\pi c}\} = \bar{c}.$$

Since $\bar{c}$ is definable, $\bar{f}$ is, by definition, also definable. Moreover, it is continuous by construction. Therefore the refined cell decomposition of $X \subset \mathbb{R}^n$ is well-based. $\square$

**1.2.16 Theorem.** *Given a compact definable subspace $X \subset \mathbb{R}^n$ with a given cell decomposition, there exists a coordinate change and a CDCP with respect to these new coordinates which refines the given cell decomposition and which has an associated CW structure.*

*Proof.* This follows immediately from Proposition 1.2.15 and Lemma 1.2.9. $\square$

## 1.3 The category Def of compact definable spaces

### 1.3.1 The definable category, Def

**1.3.2 Definition.** Let **Def** be the **definable category** which has as objects the compact definable subsets (in some fixed o-minimal expansion of $\mathbb{R}$), and as morphisms the continuous definable maps.

We make a couple of remarks here which will be important in later chapters. Firstly, we note that **Def** has initial object $\emptyset$ and terminal object $*$. Secondly, **Def** has finite limits since it clearly has finite products and equalisers. It is not immediately obvious whether or not **Def** has finite colimits (see discussion in Chapter 4, Section 4.3). However, we do have following property.

**1.3.3 Lemma.** *The category* **Def** *has pushouts along closed inclusions. In addition, taking products in* **Def** *preserves these pushouts.*

*Proof.* This will follow immediately from van den Dries' result on the existence of definable proper quotients of compact spaces by compact subspaces which is stated in Proposition 1.3.12 below. □

**1.3.4 Lemma.** *Fibre products preserve pushouts along closed inclusions in* **Def**.

*Proof.* Suppose we have $A \leftarrow B \to C$ in **Def** where $B \to C$ is a closed inclusion, and consider the pushout $A \vee_B C$. Moreover, suppose that there exists a morphism $A \vee_B C \to Y$, and that we are given another morphism $X \to Y$ in **Def**. Then there is a unique dotted morphism in the following diagram:

$$\begin{tikzcd}
B \times_Y X \arrow[r] \arrow[d] & C \times_Y X \arrow[d] \arrow[ddr, bend left] & \\
A \times_Y X \arrow[r] \arrow[ddrr, bend right] \arrow[ur, phantom, "\ulcorner"] & (A \times_Y X) \vee_{B \times_Y X} (C \times_Y X) \arrow[dr, dashed, "\exists!"] & \\
& & (A \vee_B C) \times_Y X.
\end{tikzcd}$$

Moreover, it is a bijection (since it is a bijection in **Set**). Hence, by Lemma 1.1.13, it is a homeomorphism, and therefore an isomorphism in **Def**. □

**1.3.5 Definable quotients** Definable mapping cones will be a key feature in the definition of the definable Spanier–Whitehead categories in Section 4.4. Of particular interest to us will be the (pointed) mapping cone sequences

$$X \xrightarrow{f} Y \to C_{\mathbf{Def}_*} f \to \Sigma_{\mathbf{Def}_*}(X) \to \ldots$$

In order to define constructions such as the mapping cone on a definable map and the suspension of a definable space, the notion of a quotient of a definable space by a compact subspace is required. Again, we present the key facts and we refer the reader to [vdD98, p.161] for further details.

**1.3.6 Definition.** A morphism $f : X \to Y$ in **Def** from $X$ to $Y \subseteq \mathbb{R}^n$ is **definably identifying** if $f$ is surjective, and for each definable $K \subseteq Y$ we have:

$$f^{-1}(K) \text{ is closed in } X \Rightarrow K \text{ is closed in } Y.$$

Since $f$ is continuous, this implies $f^{-1}K$ is closed in $X$ if and only if $K$ is closed in $Y$, so that $Y$ has the quotient topology.

A relation $R$ on $X \in \mathbf{Def}$ is a subset of $X \times X$. A relation is definable if it is a definable subset of $X \times X$.

Suppose $f : X \to Y$ is a morphism in **Def**. Let

$$E_f = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$$

be the kernel of $f$ (i.e. the set of points having the same image). This is a definable equivalence relation on $X$. Since $f$ is continuous, $E_f$ is closed in $X \times X$.

**1.3.7 Definition.** Suppose $X \in \mathbf{Def}$ and let $E \subseteq X \times X$ be a definable equivalence relation on $X$. **A definable quotient of** $X$ **by** $E$ is a pair $(p, Y)$ consisting of a definable set $Y \subseteq \mathbb{R}^n$, together with a definable continuous surjective map $p : X \to Y$, such that $E = E_p$, i.e. $(x_1, x_2) \in E$ if and only if $p(x_1) = p(x_2)$ for all $x_1, x_2 \in X$, and such that $p$ is definably identifying (i.e. for all definable $K \subseteq Y$, if $p^{-1}(K)$ is closed in $X$, then $K$ is closed in $Y$).

**1.3.8 Definition.** A morphism $f : X \to Y$ in **Def** from $X$ into $Y \subseteq \mathbb{R}^n$ is **definably proper** if for each definable set $K \subseteq Y$ we have:

$$K \text{ is compact in } \mathbb{R}^n \Rightarrow f^{-1}(K) \subseteq X \text{ is compact in } \mathbb{R}^m.$$

For a surjective map $f : X \to Y$, if $f$ is definably proper then it is also definably identifying.

**1.3.9 Definition.** Given $X \in \mathbf{Def}$, a **definably proper quotient of** $X$ **by** $E$ is a definable quotient $(p, Y)$ of $X$ by $E$ (as in Definition 1.3.7) where $p$ is a definably proper morphism.

**1.3.10 Proposition** ([vdD98, p.162]). *Given $X \in$ **Def**, a definable quotient $p : X \to Y$ is a quotient of $X$ by a definable equivalence relation $E$ in* **Def***: If $f : X \to W$ is a morphism in* **Def** *from $X$ into a definable set $W \subseteq \mathbb{R}^n$ such that $E \subseteq E_f$, then the unique morphism $g : Y \to W$ such that $f = g \circ p$ is a morphism in* **Def** *(i.e. $g$ is continuous and definable).*

It follows that if $(p, Y)$ and $(p', Y')$ are both definable quotients of $X$ by $E$ then we have a unique definable bijection $h : Y \to Y'$ and this bijection must be a homeomorphism [vdD98, p.162]. So, up to isomorphism, there is at most one definable quotient of $X$ by $E$. Therefore, if such a quotient exists, it is more or less unique; it is called *the* definable quotient of $X$ by $E$ and we write $Y = X/E$.

Here we note that the results above hold for all definable spaces (not necessarily compact) and continuous definable morphisms. In general, for a definable quotient of $X$ by $E$ to exist it is necessary that $E$ is closed in $X \times X$. When $X$ is compact, i.e. when $X \in \mathbf{Def}$, this is also sufficient by the following theorem.

**1.3.11 Theorem** ([vdD98, p.166]). *Suppose $X$ is some definable space and $E$ is a definable equivalence relation on $X$. Let $pr_1 : E \to X$ be the restriction of the projection map $\pi_1 : X \times X \to X$ on to the first factor. If $pr_1 : E \to X$ is proper, then $X$ has a definable quotient $X/E$ by $E$.*

To construct new definable spaces such as cones and suspensions, we need to be able to *definably collapse* a subset to a point. Such collapses are given by definable quotient spaces. The following proposition tells us that this kind of definable quotient space exists if the subspace in question is compact.

**1.3.12 Proposition** ([vdD98, p.162]). *Given $X \in \mathbf{Def}$, suppose $A \subseteq X \subseteq \mathbb{R}^m$ is a nonempty subset of $X$. Let $E_A$ be the definable equivalence relation on $X$ whose equivalence classes are the singletons $\{x\}$ with $x \in X - A$, and the set $A$. If $A$ is a compact in $\mathbb{R}^m$, then there exists a definable proper quotient of $X$ by $E_A$.*

So we can define the quotient of definable space $X$ by a compact subset $A \subseteq X$ as $X/A := X/E_A$, where $E_A$ is the definable equivalence relation on $X$ whose equivalence classes are the singletons $\{x\}$ with $x \in X - A$, and the set $A$. In particular, this means that we can consider a *definable cone* or a *definable suspension* on a compact definable space. Later on, these will also be constructed abstractly just using the core properties of the category **Def**. However, it is still instructive to give explicit constructions here.

**1.3.13 Example.** Given $X \in \mathbf{Def}$, we suppose that $X \subseteq \mathbb{R}^m$. In **Def**, the **definable cylinder on** $X$ is given by

$$\mathrm{Cyl}_{\mathbf{Def}}(X) = X \times I \subseteq \mathbb{R}^{m+1}.$$

Take $A = X \times 0 \subseteq X \times I$ to be the compact subset we wish to collapse. By Proposition 1.3.12, we know that there exists a definable proper quotient of $X \times I$ by $X \times 0$. So, in the category **Def**, the **definable cone on** $X$ is defined to be

$$C_{\mathbf{Def}}X := (X \times I)/(X \times \{0\}).$$

**1.3.14 Example.** Given an object $X$ in **Def** and taking

$$A = (X \times \{-1\}) \cup (X \times \{1\}) \subseteq X \times [-1, 1]$$

to be the subset we are interested in collapsing, the **definable suspension of** $X$ in **Def** is defined to be

$$S_{\mathbf{Def}}X := (X \times [-1,1])/A.$$

Note that we have chosen to take $X \times [-1,1]$ rather than $X \times [0,1]$ in order to simplify future verifications. However this is just a technicality; suspension in **Def** could also be taken to be $S_{\mathbf{Def}}X := (X \times I)/(X \times \{0\} \cup X \times \{1\})$.

We now look at how to attach definable spaces via definable maps. To do this, we first define the notion of a disjoint sum in **Def**.

**1.3.15 Definition.** Let $S_1, ..., S_k$ be definable sets in $\mathbb{R}^{m(1)}, ..., \mathbb{R}^{m(k)}$ for $k \geq 1$. **A disjoint sum of** $S_1, ...S_k$, written as $\bigsqcup_{i=1}^{k} S_i$, is a tuple $(h_1, ..., h_k, T)$ consisting of a definable set $T \subseteq \mathbb{R}^n$ for some $n$, together with definable maps $h_i : S_i \to T$ such that:

1. $h_i$ is a homeomorphism onto $h_i(S_i)$ and $h_i(S_i)$ is open in $T$ for $i = 1, ..., k$;

2. $T$ is the disjoint union of the sets $h_1(S_1), ..., h_k(S_k)$.

Suppose $X, Y \in \mathbf{Def}$ and $A \subseteq X$. Given a definable continuous map $f : A \to Y$ from $A$ into $Y$, we want to attach $X$ to $Y$ via $f$. Let us consider the disjoint sum $X \sqcup Y$ and $E(f)$, the smallest equivalence relation on $X \sqcup Y$ for which each $a \in A \subseteq X$ is equivalent to $f(a) \in Y$. Let $\Delta_X = \{(x,x) : x \in X\}$ and $\Delta_Y = \{(y,y) : y \in Y\}$ be the diagonals of $X$ and $Y$. We have

$$E(f) = \Delta_X \cup \Delta_Y \cup \Gamma_f \cup \Gamma_f^t \cup \{(a_1, a_2) \in A \times A \mid f(a_1) = f(a_2)\},$$

which is a definable equivalence relation where $\Gamma_f^t = \{(f(a), a) \mid a \in A\}$ is the transpose of the graph $\Gamma_f$ of $f$.

If the definable quotient of $X \sqcup Y$ by $E(f)$ exists, then we denote $X \sqcup Y/E(f)$ just by $X \sqcup_f Y$, the space obtained by attaching $X$ to $Y$ via $f$. The following lemma tells us that, once again, this kind of definable quotient space exists if the subspace in question is compact.

**1.3.16 Lemma** ([vdD98, p.165]). *Suppose $X, Y \in$ **Def** and $A \subseteq X$. If $A$ is a compact definable subset of $X$ and $f : A \to Y$ is a definable continuous map from $A$ into $Y$, then $X \sqcup_f Y$ exists as a definably proper quotient of $X \sqcup Y$ by $E(f)$.*

**1.3.17 Example.** Suppose $f : X \to Y$ is a morphism in **Def**. Assume that $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$. Consider

$$\mathrm{Cyl}_{\mathbf{Def}} X \subseteq \mathbb{R}^{m+1}$$

and the compact definable subset

$$A = X \times \{1\} \subseteq \mathrm{Cyl}_{\mathbf{Def}} X$$

together with the continuous definable map

$$\hat{f} : X \times \{1\} \to Y$$

defined by $\hat{f}(x,1) = f(x)$. Then the **definable mapping cylinder on** $f$ in **Def** is defined to be the space obtained by attaching $\mathrm{Cyl}_{\mathbf{Def}} X \subseteq \mathbb{R}^{m+1}$ to $Y \subseteq \mathbb{R}^n$ via $\hat{f}$, i.e.

$$\mathrm{Cyl}_{\mathbf{Def}} f := \mathrm{Cyl}_{\mathbf{Def}} X \sqcup_{\hat{f}} Y.$$

Now consider $C_{\mathbf{Def}} X \subseteq \mathbb{R}^{c(m)}$ The space obtained by attaching $C_{\mathbf{Def}} X$ to $Y$ via $\hat{f}$ is precisely the **definable mapping cone on** $f$ in **Def**, i.e.

$$C_{\mathbf{Def}} f := C_{\mathbf{Def}} X \sqcup_{\hat{f}} Y.$$

### 1.3.18 The underlying topology of definable spaces

**1.3.19 Definition.** Let $U : \mathbf{Def} \to \mathbf{Top}$ be the **underlying topology functor** taking each definable space to its underlying topological space and taking each definable morphism to its underlying continuous map.

In this section we show that definable cylinders, cones, etc. forget to the standard ones in **Top**. Note that given definable spaces $A \subset B$, the definable inclusion $A \to B$ induces the standard inclusion $U(A) \to U(B)$.

**1.3.20 Lemma.** *Suppose $A, B \in \mathbf{Def}$. There exists a homeomorphism*

$$U(A \times B) \xrightarrow{\sim} U(A) \times U(B).$$

*Proof.* Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be definable spaces. Then applying the functor $U$, we obtain the spaces $U(A)$ and $U(B)$, subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, which inherit the subspace topology. Considering the set of pairs

$$A \times B = \{(a,b) \mid a \in A, b \in B\} \subset \mathbb{R}^{m+n},$$

it is clear that $U(A \times B) \xrightarrow{p} U(A) \times U(B)$ is given just by the identity, i.e. $(a,b) \mapsto (a,b)$. We know that the topology on $\mathbb{R}^{m+n}$ is the product topology from $\mathbb{R}^m$ and $\mathbb{R}^n$. The underlying space of the definable product, $U(A \times B)$, has the subspace topology from $\mathbb{R}^{m+n}$, and $U(A) \times U(B)$ has the product of subspace topologies from $\mathbb{R}^m$ and $\mathbb{R}^n$.

Let $J$ and $K$ be open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. For $U(A) \times U(B)$, we can choose a basis $(J \cap A) \times (K \cap B) = (J \times K) \cap (A \times B)$. Since

$$(J \times K) \cap (A \times B) \subset U(A \times B)$$

is open, the map $p$ is continuous.

Conversely, since $\mathbb{R}^{m+n}$ has the product topology, for $U(A \times B)$ we can just take a basis consisting of open subsets of the form $J \times K$ where $J$ and $K$ are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. So $p^{-1}$ is also continuous, and hence $p$ is a homeomorphism. □

We note that this result also holds for non-compact definable spaces.

**1.3.21 Lemma.** *Suppose $B \subset A \in$ **Def**. There exists a homeomorphism*

$$\beta : U(A/B) \xrightarrow{\sim} U(A)/U(B)$$

*such that $\beta \circ U(q)$ is the topological quotient map where $q : A \to A/B$ is the definable quotient.*

*Proof.* There is an obvious bijection of the underlying sets, $U(A/B) \xrightarrow{\beta} U(A)/U(B)$. The underlying map $U(q) : U(A) \to U(A/B)$ is induced from the definable quotient map $q : A \to A/B$. The topological quotient map is $Q : U(A) \to U(A)/U(B)$, and $J \subset U(A)/U(B)$ is open if and only if $Q^{-1}(J)$ is open in $U(A)$. Then we have the following commuting diagram of sets (where we don't distinguish between the map $q$ and the induced topological map $U(q)$) :

$$\begin{tikzcd} & U(A) \arrow[dl, "q"'] \arrow[dr, "Q"] & \\ U(A/B) \arrow[rr, "\beta"'] & & U(A)/U(B). \end{tikzcd}$$

First let us suppose that $K \subset U(A/B)$ is an open subset of $U(A/B)$. Then $q^{-1}(K) \subset U(A)$ is open since $q$ is a continuous map. We know that $\beta$ is a bijection so we can write $K = \beta^{-1}(\beta K)$, so that $q^{-1}(\beta^{-1}(\beta K))$ is open in $U(A)$. Since the diagram commutes, we have $q^{-1} \circ \beta^{-1} = Q^{-1}$, thus $Q^{-1}(\beta K) \subset U(A)$ is open. The map $Q$ is the topological quotient map so

$$Q^{-1}(\beta K) \subset U(A) \text{ open } \Rightarrow \beta K \subset U(A)/U(B) \text{ open.}$$

We have shown that if $K \subset U(A/B)$ is open, then $\beta K \subset U(A)/U(B)$ is open. Thus $\beta^{-1}$ is continuous.

Let us now consider the continuous bijection $\beta^{-1} : U(A)/U(B) \to U(A/B)$. To prove that $\beta$ is a continuous map, we use the standard result which tells us that, given two topological spaces $X$ and $Y$ where $X$ is compact and $Y$ is Hausdorff, any continuous bijection $f : X \to Y$ is a homeomorphism. Working with compact definable spaces, we know that the underlying topological space of a definable space is Hausdorff. Hence both $U(A)$ and $U(B)$ are compact and Hausdorff. So we need to show that

(i) $U(A)$ compact $\Rightarrow U(A)/U(B)$ compact,

(ii) $U(A)/U(B)$ Hausdorff.

For $(i)$: Let us suppose we have a cover of $U(A)/U(B)$. This can be lifted to a cover of $U(A)$. Hence it has a finite subcover. So we deduce that $U(A)/U(B)$ must also have a finite subcover. Hence $U(A)/U(B)$ is compact.

For $(ii)$: Clearly, since $A/B$ is a definable quotient, $U(A/B)$ is Hausdorff.

Therefore, by the well-known result above, $\beta^{-1}$ is a homeomorphism. □

The following corollary is immediate.

**1.3.22 Corollary.**

1. *Given $X \in \mathbf{Def}$, there exists a homeomorphism $U(S_{\mathbf{Def}}X) \xrightarrow{\sim} SU(X)$.*

2. *Given a morphism $f : X \to Y$ in $\mathbf{Def}$, there exists a homeomorphism*

$$U(C_{\mathbf{Def}}(f)) \cong C(U(f)).$$

3. *Given a morphism $f : X \to Y$ in $\mathbf{Def}$, the canonical inclusion $Y \to C_{\mathbf{Def}}(f)$ induces*

$$U(Y) \to C(U(f))$$

*(since $C(U(f)) \cong U(C_{\mathbf{Def}}(f))$).*

4. *Given a morphism $f : X \to Y$ in $\mathbf{Def}$, the quotient map $C_{\mathbf{Def}}(f) \to S_{\mathbf{Def}}X$ induces*

$$C(U(f)) \to S(U(X))$$

*(since $C(U(f)) \cong U(C_{\mathbf{Def}}(f))$ and $S(U(X) \cong U(S_{\mathbf{Def}}X))$.*

For completeness the following lemma, which is basically Lemma 1.1.13, is restated in the setting of the definable category. It tells us than we can forget, not just to **Top** via the functor $U$, but essentially all the way to **Set**.

**1.3.23 Lemma.** *If $f : X \to Y$ is a continuous bijection in* **Def**, *then it is an isomorphism.*

*Proof.* This follows from the fact that a continuous definable bijection between compact definable spaces is a homeomorphism by Lemma 1.1.13. Hence $f$ is an isomorphism in **Def**. □

**1.3.24 Definable homotopies** The notions of o-minimal homotopies and o-minimal homotopy groups are defined and studied in [BO10]. In this section we will briefly present the key results. Again, we are fixing $\mathcal{R}$ to be an o-minimal expansion of $\mathbb{R}$ (and $\mathcal{R}^{\mathcal{SA}}$ to be the smallest o-minimal expansion of $\mathbb{R}$, i.e. the class of semialgebraic sets).

**1.3.25 Definition. A definable homotopy between** $f : A \to B$ **and** $g : A \to B$ (where $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ are definable spaces and $f, g$ are definable continuous maps) is a definable continuous map

$$H : A \times I \to B,$$

where $I = [0, 1]$, such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1)$ for all $x$ in $A$. If such a homotopy exists, we write $f \simeq_{Def} g$.

$H$ can be viewed as a "continuous" family of maps $(H_t)_{0 \leq t \leq 1}$, with $H_t : A \to B$ given by $H_t(x) = H(x, t)$, so that $f = H_0$ and $g = H_1$. Note that each slice $H_t$ is definable since its graph $\Gamma_{H_t} = \Gamma_H \cap (A \times \{t\} \times B)$ is definable.

**1.3.26 Lemma.** *Given two continuous definable maps $f$ and $g$, if $f \simeq_{Def} g$ then $U(f) \simeq U(g)$.*

*Proof.* Since a definable homotopy is simply a definable map $h : X \times I \to Y$, we have

$$U(X \times I) \cong U(X) \times U(I) \xrightarrow{U(h)} U(Y).$$

Hence the result follows immediately. □

The standard properties of classical homotopies also hold for definable homotopies. For example, the proof that definable homotopy is an equivalence relation follows the standard proof; reflexivity is evident, symmetry only requires the definable map $t \mapsto 1 - t$ for the inverse homotopy, and to show transitivity holds one only needs the definble map $t \mapsto 2t$. We also have the following notions which apply in the definable setting:

**1.3.27 Definition.** A definable set $A$ is **definably contractible to the point** $a \in A$ if there is a definable homotopy $H : A \times I \to A$ between the identity map on $A$ and the map $A \to A$ taking the constant value $a$.

Let $A' \subseteq A \subseteq \mathbb{R}^M$ be given definable subsets. We denote the inclusion of $A'$ in $A$ by $i_{A'} : A' \to A$.

**1.3.28 Definition. A definable retraction** is a continuous definable map

$$r : A \to A'$$

such that $r(x) = x$ for all $x \in A'$. A definable retraction satisfies $r \circ i_{A'} = 1_{A'}$.

**1.3.29 Definition. A definable contraction from** $A$ **to** $A'$ is a definable homotopy $H : A \times I \to A$ between $1_A$, the identity on $A$, and $i_{A'} \circ r$ for a definable retraction $r : A \to A'$. The map $r$ is then uniquely determined by $r(x) = H(x, 1)$ (and $H$ **can be called a definable contraction from** $A$ **to** $r$).

**1.3.30 Definition. A definable strong deformation retraction from** $A$ **to** $A'$ is a definable contraction $H : A \times I \to A$ between $1_A$ and $i_{A'} \circ r$ such that $H(a', t) = a'$ for all $a' \in A'$ and $t \in I$.

The following lemma by Baro and Otero, known as the o-minimal Homotopy Extension Lemma, tells us that any closed inclusion $A \subset X$ of definable spaces is a cofibration.

**1.3.31 Lemma** ([BO10, p.2], The o-minimal Homotopy Extension Lemma)**.** *Let $X$, $Z$ and $A \subseteq X$ be definable sets with $A$ closed in $X$. Let $f : X \to Z$ be a definable map and $H : A \times I \to Z$ a definable homotopy such that $H(x, 0) = f(x)$ for $x \in A$. Then there exists a definable homotopy*

$$G : X \times I \to Z$$

*such that $G(x, 0) = f(x)$ for $x \in X$, and $G|_{A \times I} = H$.*

We now consider $(X, A)$ and $(Y, B)$, two pairs of definable sets. Let $C$ be a relatively closed definable subset of $X$ and let $h : C \to Y$ be a definable map such that $h(A \cap C) \subseteq B$.

**1.3.32 Definition.** Two maps $f, g : (X, A) \to (Y, B)$ with $f|_C = g|_C$ are **definably homotopic relative to** $h$, $f \sim_h g$, if there exists a definable map

$$H : (X \times I, A \times I) \to (Y, B)$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$, and $H(x,t) = h(x)$ for all $x \in C$ and $t \in I$. Definable homotopy gives an equivalence relation $\sim_h$.

**1.3.33 Definition. The o-minimal homotopy set of $(X, A)$ and $(Y, B)$ relative to $h$** is the set

$$[(X,A),(Y,B)]_h^{\mathcal{R}} = \left\{ f \mid f : (X,A) \to (Y,B) \text{ definable in } \mathcal{R},\ f|_C = h \right\} / \sim_h .$$

For non-relative versions (i.e. for $C = \emptyset$) we omit $h$ in the above definition. We now look at the relationships between semialgebraic, o-minimal and classical homotopies.

**1.3.34 Theorem** ([BO10, p.5])**.** *Let $(X, A)$ and $(Y, B)$ be two pairs of semialgebraic sets with $X$ compact (in some $\mathbb{R}^n$). Let $C \subseteq X$ be a closed semialgebraic subset of $X$ and $h : C \to Y$ a semialgebraic map such that $h(A \cap C) \subseteq B$. If $A$ is closed in $X$, then the map*

$$\begin{aligned} \rho : [(X,A),(Y,B)]_h^{\mathcal{R}^{\mathcal{SA}}} &\to [(X,A),(Y,B)]_h^{\mathcal{R}} \\ [f] &\mapsto [f] \end{aligned}$$

*is a bijection.*

We denote a classical homotopy set by $[U(X), U(Y)]$ where $U(X)$ and $U(Y)$ are the underlying topological spaces of the semialgebraic sets $X$ and $Y$ (also called their geometric realisations).

**1.3.35 Corollary** ([BO10, p.8])**.** *If $(X, A)$ and $(Y, B)$ are two pairs of semialgebraic sets then there exists a bijection*

$$\rho : [(U(X), U(A)), (U(Y), U(B))] \to [(X,A),(Y,B)]^{\mathcal{R}}.$$

**1.3.36 The pointed definable category, $\mathbf{Def}_*$** We are particularly interested in *pointed* definable spaces.

**1.3.37 Definition.** A **pointed definable space** is a definable space $X \subseteq \mathbb{R}^m$, together with a choice of basepoint $x_0 \in X$.

**1.3.38 Definition.** Given pointed definable spaces $X$ and $Y$ with basepoints $x_0$ and $y_0$ respectively, a **pointed definable continuous map** is a basepoint-preserving definable continuous map $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$.

**1.3.39 Definition.** The category $\mathbf{Def}_*$ is the pointed definable category which has as objects the compact pointed definable subsets (in an o-minimal expansion of $\mathbb{R}$), and as morphisms the pointed definable continuous maps.

**1.3.40 Remark.** The category $\mathbf{Def}_*$ is exactly the category obtained by taking the coslice category $*/\mathbf{Def}$ so that $*$ becomes both the initial and the terminal object. Note that the limits in $\mathbf{Def}_*$ are the same as those in $\mathbf{Def}$.

All the results of the previous section on definable quotients in $\mathbf{Def}$ also hold for $\mathbf{Def}_*$. In particular, we can define reduced versions of the definable cone, suspension, mapping cylinder and mapping cone. We fix the interval object $I = [0, 1]$ to have basepoint at 0 in $\mathbf{Def}_*$.

**1.3.41 Example.** Given $X \in \mathbf{Def}_*$, consider the subset $A = \{x_0\} \times I$ of $X \times I$. Since $A$ is compact, by Proposition 1.3.12, we know that there exists a definable proper quotient of $X \times I$ by $A$. The **reduced definable cylinder on** $X$ in $\mathbf{Def}_*$ is then defined as

$$\mathrm{Cyl}_{\mathbf{Def}_*} = (X \times I)/(\{x_0\} \times I).$$

Now take $A = (X \times \{0\}) \cup (\{x_0\} \times I)$ to be the compact subset of $X \times I$ that we wish to collapse. By Proposition 1.3.12, there exists a definable proper quotient of $X \times I$ by $(X \times 0) \cup (\{x_0\} \times I)$. So, in the category $\mathbf{Def}_*$, the **reduced definable cone on** $X$ is defined to be

$$C_{\mathbf{Def}_*} X := (X \times I)/((X \times \{0\}) \cup (\{x_0\} \times I)).$$

**1.3.42 Example.** Similarly, given $X \in \mathbf{Def}_*$ and taking

$$A = (X \times \{-1\}) \cup (X \times \{1\}) \cup (\{x_0\} \times [-1, 1])$$

as the compact subset of $X \times [-1, 1]$ to be collapsed, the **reduced definable suspension of** $X$ in $\mathbf{Def}_*$ can be defined to be

$$\Sigma_{\mathbf{Def}_*}(X) := (X \times [-1, 1])/((X \times \{-1\}) \cup (X \times \{1\}) \cup (\{x_0\} \times [-1, 1]).$$

**1.3.43 Example.** Suppose $X, Y \in \mathbf{Def}_*$ and $f : X \to Y$ is a morphism in $\mathbf{Def}_*$. Consider $\mathrm{Cyl}_{\mathbf{Def}_*} X$ and the compact definable subset $A = X \times \{1\} \subseteq \mathrm{Cyl}_{\mathbf{Def}_*} X$ together with the pointed definable continuous map $\hat{f} : X \times \{1\} \to Y$ defined by $\hat{f}(x, 1) = f(x)$. By Lemma 1.3.16, $\mathrm{Cyl}_{\mathbf{Def}_*} X \sqcup_f Y$ exists as a definably proper quotient of $\mathrm{Cyl}_{\mathbf{Def}_*} X \sqcup Y$ by $E(f)$ (the smallest equivalence relation on $\mathrm{Cyl}_{\mathbf{Def}_*} X \sqcup Y$ for which each $a \in A \subseteq \mathrm{Cyl}_{\mathbf{Def}_*} X$ is equivalent to $f(a) \in Y$).

Then the **reduced definable mapping cylinder on** $f$ in $\mathbf{Def}_*$ is defined to be the space obtained by attaching $\mathrm{Cyl}_{\mathbf{Def}_*} X$ to $Y$ via $\hat{f}$, i.e.

$$\mathrm{Cyl}_{\mathbf{Def}_*} f := \mathrm{Cyl}_{\mathbf{Def}_*} X \sqcup_{\hat{f}} Y.$$

Now consider $C_{\mathbf{Def}_*}X$. In a similar fashion, the space obtained by attaching $C_{\mathbf{Def}_*}X$ to $Y$ via $\hat{f}$ is defined to be the **reduced definable mapping cone on** $f$ in $\mathbf{Def}_*$, i.e.

$$C_{\mathbf{Def}_*}f := C_{\mathbf{Def}_*}X \sqcup_{\hat{f}} Y.$$

We can now show that, given a morphism $f : X \to Y$ in $\mathbf{Def}_*$, the reduced definable suspensions and mapping cones give rise to *definable mapping cone sequences.*

**1.3.44 Lemma.** *Given* $f : X \to Y$ *in* $\mathbf{Def}_*$*, there exists a* **definable mapping cone sequence** *in* $\mathbf{Def}_*$ *given by*

$$X \xrightarrow{f} Y \to C_{\mathbf{Def}_*}f \to \Sigma_{\mathbf{Def}_*}(X) \to \dots \tag{1.3.1}$$

*Proof.* Suppose $f : X \to Y$ is a morphism in $\mathbf{Def}_*$. Clearly there is a continuous definable inclusion $Y \to C_{\mathbf{Def}_*}(f)$ in $\mathbf{Def}_*$. We can take another definable equivalence relation, $C_{\mathbf{Def}_*}(f)/E_Y$, on the mapping cone $C_{\mathbf{Def}_*}f$, where

$$E_Y = \{(y, y') \mid y, y' \in Y \subset C_{\mathbf{Def}_*}(f)\}$$

(so that all pairs of points in $Y$ get identified). We note that we are in fact identifying $Y$ with its image in the disjoint union (i.e. $h(Y)$ as in Definition 1.3.15). Recall that $C_{\mathbf{Def}_*}X := (X \times I)/((X \times \{0\}) \cup (\{x_0\} \times I))$ and

$$\Sigma_{\mathbf{Def}_*}(X) := (X \times [-1, 1])/((X \times \{-1\}) \cup (X \times \{1\}) \cup (\{x_0\} \times [-1, 1])).$$

There exists an obvious definable homeomorphism $X \times [-1, 1] \xrightarrow{\sim} X \times I$ which induces

$$(X\times[-1, 1])/((X\times\{-1\})\cup(\{x_0\}\times[-1, 1])) \xrightarrow{\ \sim\ } (X\times I)/((X\times\{0\})\cup(\{x_0\}\times I))$$

where $(X \times I)/((X \times \{0\}) \cup (\{x_0\} \times I)) = C_{\mathbf{Def}_*}X$. The inclusion

$$C_{\mathbf{Def}_*}X \to C_{\mathbf{Def}_*}f$$

induces $C_{\mathbf{Def}_*}X/X \times \{1\} \xrightarrow{\sim} C_{\mathbf{Def}_*}f/E_Y$. Then the definable homeomorphism

$$\Sigma_{\mathbf{Def}_*}(X) \xrightarrow{\sim} C_{\mathbf{Def}_*}X/(X \times \{1\})$$

gives the morphism $\Sigma_{\mathbf{Def}_*}(X) \xrightarrow{\sim} C_{\mathbf{Def}_*}f/E_Y$. Thus the required sequence of definable maps, (1.3.1), exists. $\square$

The underlying topology functor $U : \mathbf{Def} \to \mathbf{Top}$ (as in Definition 1.3.19) induces a functor $U : \mathbf{Def}_* \to \mathbf{Top}_*$ in a natural way. Therefore the results on the underlying topology of **Def** also hold $\mathbf{Def}_*$. We summarise these properties in the lemma below.

**1.3.45 Lemma.**

1. *Suppose $X, Y \in \mathbf{Def}_*$. There exists a homeomorphism*
$$U(X \times Y) \xrightarrow{\sim} U(X) \times U(Y).$$
2. *Suppose $A \subset X \in \mathbf{Def}_*$. There exists a homeomorphism*
$$\beta : U(X/A) \xrightarrow{\sim} U(X)/U(A)$$
*such that $\beta \circ U(q)$ is the topological quotient map where $q : X \to X/A$ is the definable quotient.*
3. *Given $X \in \mathbf{Def}_*$, there exists a homeomorphism $U(\Sigma_{\mathbf{Def}_*}(X)) \xrightarrow{\sim} \Sigma(U(X))$.*
4. *Given a morphism $f : X \to Y$ in $\mathbf{Def}_*$, there exists a homeomorphism*
$$U(C_{\mathbf{Def}_*}(f)) \cong C(U(f)).$$
5. *Given a morphism $f : X \to Y$ in $\mathbf{Def}_*$, the canonical inclusion $Y \to C_{\mathbf{Def}_*}(f)$ induces*
$$U(Y) \to C(U(f))$$
*(since $C(U(f)) \cong U(C_{\mathbf{Def}_*}(f))$).*
6. *Given a morphism $f : X \to Y$ in $\mathbf{Def}_*$, the quotient map*
$$C_{\mathbf{Def}_*}(f) \to \Sigma_{\mathbf{Def}_*}(X)$$
*induces*
$$C(U(f)) \to \Sigma(U(X))$$
*(since $C(U(f)) \cong U(C_{\mathbf{Def}_*}(f))$ and $\Sigma(U(X)) \cong U(\Sigma_{\mathbf{Def}_*}(X))$).*

**1.3.46 Remark.** Of course, Theorem 1.2.16 also holds for *pointed* compact definable spaces. So objects in **Def** and $\mathbf{Def}_*$ can be given CW-structures in a natural way, as discussed in Section 1.2.

The required notion of homotopy in $\mathbf{Def}_*$ is **pointed definable homotopy**. Consider the definition of a relative definable homotopy between maps

$$f, g : (X, A) \to (Y, B)$$

such that $f|_C = g|_C$ in Definition 1.3.32 where $C$ is a relatively closed definable subset of $X$ and $h : C \to Y$ is a definable map such that $h(A \cap C) \subseteq B$. Let $(X, A) = (X, x_0)$ and $(Y, B) = (Y, y_0)$ be two objects in $\mathbf{Def}_*$, and take $C = \{x_0\}$.

**1.3.47 Definition.** The morphisms $f, g : (X, x_0) \to (Y, y_0)$ in $\mathbf{Def}_*$ with

$$f|_{x_0} = g|_{x_0}$$

are definably homotopic relative to $h$ if there exists a definable map

$$H : (X \times I, x_0 \times I) \to (Y, y_0)$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$, and $H(x_0, t) = h(x_0)$ for all $t \in I$, and with $H(x_0, t) = f(x_0) = g(x_0) = y_0$ for all $t \in I$). The morphism $H$ is then said to be a **pointed definable homotopy between** $f$ **and** $g$. In other words, a pointed definable homotopy is just a definable homotopy relative to the basepoints (i.e. constant at the basepoints).

We now consider o-minimal homotopy groups.

**1.3.48 Definition.** Let $(X, x_0)$ be an object in $\mathbf{Def}_*$. Then $\pi_0(X, x_0)^{\mathcal{R}}$ is defined to be the set of definably connected components of $X$. **The o-minimal homotopy group of dimension** $n$**,** for $n \geq 1$, is defined to be the set

$$\pi_n(X, x_0)^{\mathcal{R}} = [(I^n, \partial I^n), (X, x_0)]^{\mathcal{R}}.$$

Just as in the classical case, we define a group operation in the o-minimal homotopy groups via the usual sum of maps for $n \geq 1$. For $n \geq 2$ these groups are abelian. Given a definable map between definable pointed sets, we define the induced map in homotopy by the usual composition. This induced map will be a homomorphism in the case where we have a group structure. One can also check that with these definitions of o-minimal homotopy group and induced map, the o-minimal homotopy groups are covariant functors. Full details are found in [BO10]. We give an alternative approach using our axiomatic construction in Chapter 3.

$\mathbf{Def}_*$ is the pointed definable category with objects the pointed compact definable subsets in some fixed o-minimal expansion of $\mathbb{R}$. Consider the subcategory $\mathbf{Def}_*^{\mathcal{R}^{\mathcal{SA}}} \subset \mathbf{Def}_*$ where the o-minimal expansion of $\mathbb{R}$ in which the pointed compact subsets are definable is the class $\mathcal{R}^{\mathcal{SA}}$ of semialgebraic sets. We have the following relationship between the semialgebraic and the o-minimal homotopy groups:

**1.3.49 Theorem** ([BO10, p.9]). *For every* $(X, x_0) \in Def_*^{\mathcal{R}^{\mathcal{SA}}}$ *and every* $n \geq 1$*, the map* $\rho : \pi_n(X, x_0)^{\mathcal{R}^{\mathcal{SA}}} \to \pi_n(X, x_0)^{\mathcal{R}}$ *given by* $[f] \mapsto [f]$*, is a natural isomorphism.*

Then the following corollary gives the relationship between the classical and the o-minimal homotopy groups.

**1.3.50 Corollary** ([BO10, p.10]). *Let* $(X, x_0)$ *be a pointed semialgebraic set. Then there exists a natural isomorphism between the classical homotopy group* $\pi_n(U(X), x_0)$ *and the o-minimal homotopy group* $\pi_n(X, x_0)^{\mathcal{R}}$ *for every* $n \geq 1$*.*

## 1.4 The constructible functions

**1.4.1 Definition.** Suppose $X$ is a definable set. **The (bounded) constructible functions**, $CF(X)$, are the subset of the bounded maps $f : X \to \mathbb{Z}$ such that $f^{-1}(n)$ is definable for each $n \in \mathbb{Z}$. We say that a constructible function $f$ is **compactly supported** if the closure $\overline{X - f^{-1}(0)}$ is compact.

**Examples:**

(i) The function $f : \mathbb{R} \to \mathbb{Z},\ x \mapsto \begin{cases} 1 & \text{for} \quad x = 0, \\ 0 & \text{for} \quad x \neq 0 \end{cases}$ is constructible, since

$$f^{-1}(0) = \mathbb{R} - \{0\},\ f^{-1}(1) = \{0\} \text{ and } f^{-1}(k) = \emptyset \text{ for } k \in \mathbb{Z},\ k \neq 0, 1.$$

(ii) The function

$$f : \mathbb{R} \to \mathbb{Z} : x \mapsto \begin{cases} x & \text{for } x \in \mathbb{Z}, \\ 0 & \text{otherwise} \end{cases}$$

is not constructible since $f^{-1}(0) = \mathbb{R} - \mathbb{Z}$ has no finite cell decomposition and so is not definable.

(iii) The function $f : \mathbb{R} \to \mathbb{Z} : x \mapsto \lfloor x \rfloor$ is constructible since $f^{-1}(n) = [n, n+1)$ is definable for each $n \in \mathbb{Z}$.

(iv) The function

$$f : \mathbb{R} \to \mathbb{Z} : x \mapsto \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & x \notin \mathbb{Q} \end{cases}$$

is not constructible since neither $f^{-1}(0)$, nor $f^{-1}(1)$, are definable.

The most important examples of constructible functions are the indicator functions of definable sets. We recall that for a definable subspace $A \subset X$, **the**

**indicator function** of $A$ is

$$1_A : X \to \mathbb{Z} : \ x \mapsto \begin{cases} 1 & \text{for} \quad x \in A, \\ 0 & \text{for} \quad x \notin A. \end{cases}$$

The indicator function of a definable subspace $A \subset X$ is constructible since $f^{-1}(0) = X - A$, $f^{-1}(1) = A$ and $f^{-1}(k) = \emptyset$ for $k \neq 0, 1$.

One crucial fact about the bounded constructible functions is that they can be written as a weighted sum of indicator functions:

**1.4.2 Proposition.** *Any bounded constructible function, $f : X \to \mathbb{Z}$, is a linear combination of indicator functions of fibres:*

$$f = \sum_{n \in \mathbb{Z}} n 1_{f^{-1}(n)}.$$

*In fact, since $f^{-n}$ can be written as a finite union of definable cells, any bounded constructible function can be written as a linear combination of indicator functions of definable cells.*

*Proof.* This is obvious since we are considering bounded constructible functions and we have $x \in f^{-1}(f(x))$ for any $x \in X$. □

**1.4.3 Proposition.** *Let $X$ be a definable space. The bounded constructible functions $CF(X)$ form a ring under point-wise addition and multiplication of functions.*

*Proof.* By Proposition 1.4.2, it is enough to show that the sum and product of indicator functions are constructible. This is clear since

$$1_A + 1_B = 1_{A \cup B} + 1_{A \cap B}$$

and

$$1_A \cdot 1_B = 1_{A \cap B},$$

and because both $A \cup B$ and $A \cap B$ are definable whenever $A$ and $B$ are definable. □

The ring of constructible functions is both covariantly and contravariantly functorial.

**1.4.4 Definition.** Given a continuous definable function $\beta : X \to Y$, the pull-back

$$\beta^* : CF(Y) \to CF(X)$$

is defined by

$$(\beta^* f)(x) = f(\beta(x))) = (f \circ \beta)(x)$$

where $f \in CF(Y)$. Since $(f \circ \beta)^{-1}(n) = \beta^{-1}(f^{-1}(n))$, it is easy to verify that $\beta^* f \in CF(X)$.

**1.4.5 Definition.** Suppose $\beta : X \to Y$ is a continuous definable map and $A \subset X$. For an indicator function $1_A$ we define the pushforward

$$(\beta_* 1_A)(y) := \chi_c(A \cap \beta^{-1}(y)).$$

To verify that $\beta_* 1_A \in CF(Y)$, we apply the Trivialisation Theorem 1.1.20 to $\beta|_A$. For a general constructible function we extend linearly, by defining

$$\beta_*(f)(y) = \sum_{n \in \mathbb{Z}} n \chi_c(f^{-1}(n) \cap \beta^{-1}(y)).$$

Since $f$ is bounded, this is a finite sum, hence also constructible.

**1.4.6 Remark.** Given a continuous definable map $\beta : X \to \mathrm{pt}$ and a constructible function $f \in CF(X)$, the **Euler integral** of $f$ is the pushforward to a point: $\int_X f \; d\chi_c := \beta_*(f)$. This terminology is chosen because the operator $\int_X d\chi_c$ is analogous to ordinary integrals in many respects. We refer the reader to [CGR12] for further details.

# Chapter 2

# The classical Spanier–Whitehead category

Motivated by the Frendenthal Suspension Theorem, [Fre38], and in the hope of isolating stable phenomena, the original Spanier–Whitehead category, introduced by Spanier and Whitehead in [SW62], consists of pointed CW-complexes and morphisms which are colimits over homotopy classes of continuous functions between suitably high suspensions. This original category is a triangulated category in the sense developed by Jean-Louis Verdier in [Ver96]. However, it does not have all coproducts, and the coproducts which exist in the homotopy category of pointed CW-complexes are not preserved under the canonical functor to the Spanier–Whitehead category. It was noticed later (by Whitehead in [Whi65]) that actually the more useful version of this category is its full subcategory of finite CW-complexes, which possesses all coproducts, and in particular is the starting point for the development of stable homotopy theory. In more recent studies of the subject, such as [Sch10], it has become conventional to take the Spanier–Whitehead category to be this latter category with objects the finite pointed CW-complexes. We will follow this custom and fix the Spanier–Whitehead category, $\mathbf{SW}(\mathbf{CW}_*)$, to have as objects the finite pointed CW-complexes and as morphisms the colimits over homotopy classes of continuous functions between suitably high suspensions, i.e. stabilised versions of homotopy classes. Since the colimit is obtained by an iteration of suspensions, by Freudenthal's Suspension Theorem, it must actually be attained at some finite stage. Thus, two finite pointed CW-complexes become isomorphic in the SW-category if and only if they become homotopy equivalent after some finite number of suspensions.

The SW-category $\mathbf{SW}(\mathbf{CW}_*)$ has a shift functor which is isomorphic to the (reduced) suspension functor and which accounts for the existence of formal de-suspensions of spaces. The morphism sets in the SW-category are naturally abelian groups since every object can be written as a double suspension. Hence $\mathbf{SW}(\mathbf{CW}_*)$ is an additive category. There is a collection of distinguished triangles in $\mathbf{SW}(\mathbf{CW}_*)$ which are given by the mapping cone sequences and their suspensions or de-suspensions. A proof that the axioms of a triangulated category hold for the Spanier–Whitehead category is provided by Margolis in [Mar83]. However, Margolis uses the original Spanier–Whitehead category construction and doesn't impose that objects be finite (as in [SW62]). Our notion of the Spanier–Whitehead category, $\mathbf{SW}(\mathbf{CW}_*)$, is denoted by $\mathbf{SW}_f$ in [Mar83]. We also note that Margolis verifies the axioms of a triangulated category as given by Dold and Puppe in [DP61]. Although these are very similar to those given by Jean-Louis Verdier, they do not impose the octahedral axiom. Thus Margolis proof does not included a verification of the octahedral axiom (in fact, we were unable to find a proof in the literature that the SW-category is triangulated which does explicitly verify the octahedral axiom).

We will be interested in constructing various (definable) analogues of the SW-category. Each of these, being a triangulated category, will have a corresponding Grothendieck group. Of particular interest to us will be the definable relative version of the SW-category where objects are compact definable spaces equipped with maps from and to some fixed base-space. Understanding the Grothendieck group of this category will be one of our main aims.

In this chapter we seek to give an overview of the Spanier–Whitehead category construction and its triangulated structure. In Section 2.1 we present the modern-day axioms of a triangulated category as in, for example, [HJ10]. Hereafter, we closely follow Margolis' approach to the construction of the Spanier–Whitehead category in [Mar83]. Section 2.2 provides the definition and basic properties of the category of finite pointed CW-complexes, and Section 2.3 looks at the corresponding homotopy category. This latter category, although not quite triangulated itself, is a stepping-stone category in the construction of the triangulated Spanier–Whitehead category of finite pointed CW-complexes. Finally, in Section 2.4, we formally define the Spanier–Whitehead category and set out Margolis' proof that it is a triangulated category.

## 2.1 A brief introduction to triangulated categories

**2.1.1 Definition.** A category $\mathbf{A}$ is an **additive category** if

**(A1)** for any $X, Y \in \mathrm{Ob}(\mathbf{A})$, the set $\mathrm{Hom}_{\mathbf{A}}(X, Y)$ is an abelian group, and composition of morphisms is bilinear;

**(A2)** $\mathbf{A}$ has a zero object, 0;

**(A3)** there exists a binary coproduct in $\mathbf{A}$.

**2.1.2 Definition.** An **additive functor** $T$ is a functor between additive categories such that for any pair $X, Y$, the morphism

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}(T(X),T(Y))$$

is a homomorphism of abelian groups.

Suppose $\mathbf{T}$ is an additive category. Let $\Sigma : \mathbf{T} \to \mathbf{T}$ be an additive functor which is an automorphism. In other words, there exists a functor $\Sigma^{-1} : \mathbf{T} \to \mathbf{T}$ such that $\Sigma \circ \Sigma^{-1} = id_{\mathbf{T}}$ and $\Sigma^{-1} \circ \Sigma = id_{\mathbf{T}}$.

**2.1.3 Definition.** A **triangle** in $\mathbf{T}$ is a sequence $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$, where $X, Y, Z \in \mathrm{Ob}(\mathbf{T})$ and $u, v, w \in \mathrm{Hom}(\mathbf{T})$. A **morphism of triangles** in $\mathbf{T}$ is a triple of morphisms $(f, g, h) \in \mathrm{Hom}(\mathbf{T})$ which make the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma(X) \arrow[d, "\Sigma(f)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma(X')
\end{tikzcd}$$

**2.1.4 Definition. A triangulated category** is an additive category $\mathbf{T}$ together with an additive automorphism $\Sigma$ (called the shift or translation functor), and a collection of distinguished triangles, $\Delta$, which satisfy the following axioms:

**TR.1** If a triangle is isomorphic to a distinguished triangle, then it is a distinguished triangle itself.

**TR.2** For all $X$ in $\mathrm{Ob}(\mathbf{T})$, the triangle $0 \xrightarrow{i_X} X \xrightarrow{id_X} X \to 0$ is in $\Delta$.

**TR.3** For all $f$ in $\mathrm{Hom}_{\mathbf{T}}(X,Y)$, there exists a distinguished triangle of the form $X \xrightarrow{f} Y \to Z \to \Sigma(X)$.

**TR.4** The triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ is in $\Delta$ if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \xrightarrow{-\Sigma u} \Sigma(Y)$ is in $\Delta$.

**TR.5** Given distinguished triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma(X')$ and morphisms $f$ and $g$ such that $gu = u'f$, then there exists some morphism $h$ (not necessarily unique) making all the squares in the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & \Sigma(X) \arrow[d, "\Sigma(f)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma(X')
\end{tikzcd}$$

**TR.6** (The octahedral axiom) Given two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbf{T}$ the distinguished triangles on $f, g$, and on the composition morphism $gf$ can be formed:

$$X \xrightarrow{f} Y \xrightarrow{f_1} Z' \xrightarrow{f_2} \dots ,$$

$$Y \xrightarrow{g} Z \xrightarrow{g_1} X' \xrightarrow{g_2} \dots ,$$

$$X \xrightarrow{gf} Z \xrightarrow{(gf)_1} Y' \xrightarrow{(gf)_2} \dots .$$

Then there exists a distinguished triangle

$$Z' \xrightarrow{\gamma} Y' \xrightarrow{\beta} X' \xrightarrow{\alpha} \dots ,$$

such that the triangles with an odd number of solid arrows in the following diagram commute:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[rrrr, bend left, "gf"] & & Y \arrow[rr, "g"] \arrow[dl, "f_1"] & & Z, \arrow[dl, "g_1"] \arrow[ddll, bend left, "(gf)_1"] \\
& Z' \arrow[ul, dashed, "f_2"] \arrow[dr, "\exists!\gamma"] & & X' \arrow[ll, dashed, "\alpha"] \arrow[ul, dashed, "g_2"] & \\
& & Y' \arrow[uull, dashed, bend left, "(gf)_2"] \arrow[ur, "\exists!\beta"] & &
\end{tikzcd}$$

i.e.

$g_1 = \beta \circ (gf)_1,$

$f_2 = (gf)_2 \circ \gamma,$

$\alpha = \Sigma(f_1) \circ g_2,$

and such that $g_2 \circ \beta = (\Sigma f) \circ (gf)_2$ and $\gamma \circ f_1 = (gf)_1 \circ g.$

## 2.2 The pointed category of finite CW-complexes, $\mathbf{CW}_*$

Let $\mathbf{CW}_*$ be the category of finite pointed CW-complexes and basepoint preserving cellular maps. We recall the definitions of the wedge sum, $\vee$, and smash product, $\wedge$, of two pointed CW-complexes $(X, x_0)$ and $(Y, y_0)$ :

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y,$$

and

$$X \wedge Y = X \times Y / X \vee Y.$$

**2.2.1 Definition.** The **(reduced) cylinder functor** $M_* : \mathbf{CW}_* \to \mathbf{CW}_*$ is defined by

$$M_*(X) = \mathrm{Cyl}_*(X) := (\{*\} \sqcup I) \wedge X,$$

where $\{*\} \sqcup I$ is the unit interval $[0,1]$ with disjoint basepoint $\{*\}$. On morphisms $f : X \to Y$ in $\mathbf{CW}_*$ we have

$$\mathrm{Cyl}_*(X) = (\{*\} \sqcup I) \wedge X \xrightarrow{M_*(f)} (\{*\} \sqcup I) \wedge Y = \mathrm{Cyl}_*(Y).$$

**2.2.2 Definition.** The **(reduced) cone functor** $\mathrm{Cone}_* : \mathbf{CW}_* \to \mathbf{CW}_*$ is defined by

$$\mathrm{Cone}_*(X) = \mathrm{C}_*(X) := I_0 \wedge X,$$

where $I_0$ denotes the unit interval $[0,1]$ with basepoint $0 \in [0,1]$. On morphisms $f : X \to Y$ in $\mathbf{CW}_*$ we have

$$\mathrm{C}_*(X) = I_0 \wedge X \xrightarrow{\mathrm{Cone}_*(f)} I_0 \wedge Y = \mathrm{C}_*(Y).$$

**2.2.3 Remark.** We use the notation $M_*$ for the reduced cylinder functor on $\mathbf{CW}_*$ rather than $\mathrm{Cyl}_*$ in order to distinguish between the cylinder functor on a morphism $f : X \to Y$ in $\mathbf{CW}_*$, denoted $M_*(f)$, which takes $C_*(X)$ to $C_*(Y)$, and the mapping cylinder on a morphism $f \in \mathbf{CW}_*$, denoted $\mathrm{Cyl}_*(f)$, which is an object in $\mathbf{CW}_*$ that will be defined below. Similarly for the reduced cone functor on $f$, denoted $\mathrm{Cone}_*(f)$, and the reduced mapping cone on $f$ which will be denoted by $C_*(f)$.

**2.2.4 Definition.** The **(reduced) suspension functor** $\Sigma_* : \mathbf{CW}_* \to \mathbf{CW}_*$ is defined by

$$\Sigma_*(X) = S^1 \wedge X,$$

where $S^1$ is the unit circle with basepoint $s^* \in S^1$. On morphisms $f : X \to Y$ in $\mathbf{CW}_*$ we have

$$\Sigma_*(X) = S^1 \wedge X \xrightarrow{\Sigma_*(f)} S^1 \wedge Y = \Sigma_*(Y).$$

**2.2.5 Definition.** Given $f : X \to Y$ in $\mathbf{CW}_*$, the **(reduced) mapping cylinder** of $f$ is defined to be

$$\mathrm{Cyl}_*(f) = ([0,1] \times X) \sqcup Y / \sim$$

where $(1, x) \sim f(x)$ for all $x \in X$ and $(s, x_0) \sim (t, x_0)$ for all $s, t \in [0,1]$. We can also write

$$\mathrm{Cyl}_*(f) = \mathrm{Cyl}_*(X) \sqcup_f Y.$$

**2.2.6 Definition.** Given $f : X \to Y$ in $\mathbf{CW}_*$, the **reduced mapping cone** of $f$ is defined to be

$$\mathrm{C}_*(f) = ([0,1] \times X) \sqcup Y/\sim$$

where $(1,x) \sim f(x)$ for all $x \in X$ and $(0,x_1) \sim (0,x_2)$ for all $x_1, x_2 \in X$ and $(s,x_0) \sim (t,x_0)$ for all $s,t \in [0,1]$. Equivalently, we have $\mathrm{C}_*(f) = \mathrm{C}_*(X) \sqcup_f Y$ or $\mathrm{C}_*(f) = \mathrm{Cyl}_*(f)/\sim$ where $\sim$ is the additional identification $(0,x_1) \sim (0,x_2)$ for all $x_1, x_2 \in X$ which collapses $\{0\} \times X$ down to a point.

## 2.3 The homotopy category of pointed finite CW-complexes, $\mathbf{CW}_*^{\mathbf{h}}$

**2.3.1 Definition.** Given two morphisms $f, g \in \mathrm{Hom}_{\mathbf{CW}_*}\big(X,Y\big)$, we say that $f$ and $g$ are **(based) homotopic** (relative to $x_0$) if there exists a map $h : \mathrm{Cyl}_*(X) \to Y$ in $\mathbf{CW}_*$ such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$ for all $x \in X$ and $h(t,x_0) = f(x_0) = g(x_0) = y_0$ for all $t \in I = [0,1]$. We write $f \simeq g$.

**2.3.2 Proposition** ([Mau96], p.28)**.** *Based homotopy is an equivalence relation on* $\mathrm{Hom}_{\mathbf{CW}_*}\big(X,Y\big)$.

**2.3.3 Definition.** A morphism $f : X \to Y$ in $\mathbf{CW}_*$ is a **homotopy equivalence** if there exists a morphism $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. The objects $X$ and $Y$ in $\mathbf{CW}_*$ are then said to be **homotopy equivalent**, $X \simeq Y$.

**2.3.4 Definition. The homotopy category of pointed CW-complexes, $\mathbf{CW}_*^{\mathbf{h}}$**, is the category whose objects are the same as those of $\mathbf{CW}_*$, but whose morphisms are given by morphisms of $\mathbf{CW}_*$ modulo the homotopy relation, i.e.

$$\mathrm{Ob}(\mathbf{CW}_*^{\mathbf{h}}) = \mathrm{Ob}(\mathbf{CW}_*),$$

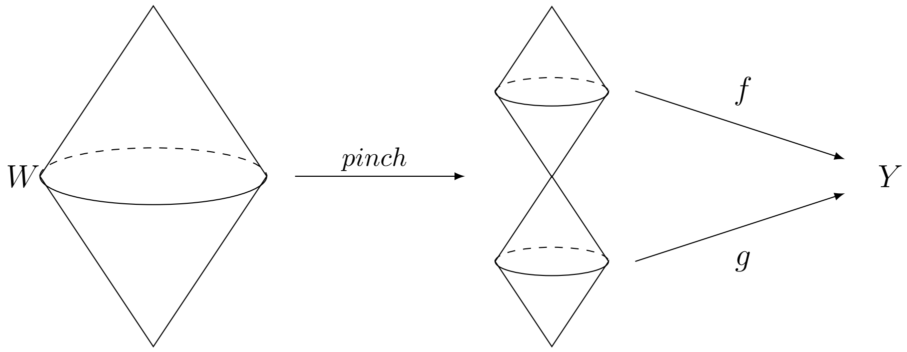
$$\mathrm{Hom}_{\mathbf{CW}_*^{\mathbf{h}}}\big(A,B\big) = \mathrm{Hom}_{\mathbf{CW}_*}\big(A,B\big)/\simeq$$

where $\simeq$ denotes the homotopy relation.

We simplify notation slightly by writing $[X,Y] := \mathrm{Hom}_{\mathbf{CW}_*^{\mathbf{h}}}\big(X,Y\big)$. Given $f \in [X,Y]$, we denote its class by $[f]$. So isomorphism inside $\mathbf{CW}_*^{\mathbf{h}}$ is *homotopy equivalence*. The homotopy category $\mathbf{CW}_*^{\mathbf{h}}$ is a key stepping stone for constructing the Spanier–Whitehead category. In particular because $\mathbf{CW}_*^{\mathbf{h}}$ is almost a triangulated category itself, but not quite since it fails to be additive. By definition, the properties and structure of $\mathbf{CW}_*$ all carry over to $\mathbf{CW}_*^{\mathbf{h}}$. The wedge sum is the coproduct inside $\mathbf{CW}^{\mathbf{h}}_{*}$. The smash product and all the other geometric constructions make sense inside the homotopy category. $\mathbf{CW}^{\mathbf{h}}_{*}$ has additional properties which we will now present.

We begin by noting that in general the morphisms (pointed homotopy classes), $[X, Y]$, take values in the category of pointed sets.

If $X = \Sigma_*(W)$ for some $W \in \mathbf{CW}^{\mathbf{h}}_{*}$, then $[X, Y] = [\Sigma_*(W), Y]$ takes values in the category of groups. The group structure of $[\Sigma_*(W), Y]$ is defined in a natural way: given $f, g \in [\Sigma_*(W), Y]$, the operation $f + g$ is taken to be represented by the composite $(f \vee g) \circ \phi : \Sigma_*(W) \to \Sigma_*(W) \vee \Sigma_*(W) \to Y$ where $\phi : \Sigma_*(W) \to \Sigma_*(W) \vee \Sigma_*(W)$ is the map which pinches the 'equator' $\{0\} \times W$ of $\Sigma_*(W)$ down to a single point, as illustrated below.




$$\Sigma(W) \xrightarrow{\quad pinch \quad} \Sigma_*(W) \vee \Sigma_*(W) \xrightarrow{\quad f \vee g \quad} Y.$$

The wedge sum of the two functions, $f \vee g : \Sigma_*(W) \vee \Sigma_*(W) \to Y$, makes sense here since $f, g$ are basepoint-preserving maps with $f(x_0) = g(x_0) = y_0$, i.e. they agree at $x_0$ which is precisely the point which is common to the wedge sum of the underlying spaces.

Letting $W = S^0$ we have $\Sigma_*(W) \cong S^1$ and the above describes the group structure on the fundamental group $[S^1, Y] = \pi_1(Y, y_0)$.

We know that the homotopy groups $[S^n, Y] = \pi_n(Y, y_0)$ are abelian for $n \geq 2$. Similarly, if $X$ is a double suspension, $X = \Sigma_*^2(W) = \Sigma_*(\Sigma_*(W))$, then the group structure becomes abelian and $[\Sigma_*^2(W), Y]$ takes values in the category of abelian groups.

**2.3.5 Proposition** ([Mar83], p.4)**.** *Given a morphism $f \in \mathrm{Hom}_{\mathbf{CW}_*}\big(X, Y\big)$ we have morphisms $[\eta_f] \in [Y, \mathrm{C}_*(f)]$ and $[\kappa_f] \in [\mathrm{C}_*(f), \Sigma(X)]$. Let $\eta_f$ be the representative of $[\eta_f]$ given by the standard inclusion map of $Y$ into to the mapping cone of $f$. Let $\kappa_f$, given by $(t, x) \mapsto (t, x)$ and $y \mapsto *$, be the representative of $[\kappa_f]$. Then the sequence of maps*

$$X \xrightarrow{f} Y \xrightarrow{\eta_f} \mathrm{C}_*(f) \xrightarrow{\kappa_f} \Sigma_*(X)$$

*is* **coexact** *at* $Y$ *and* $\mathrm{C}_*(f)$ *and is called* **the mapping sequence of** $f$*.*

**2.3.6 Remark.** In general, a sequence of morphisms $P \to Q \to R$ is said to be *coexact* if for all objects $W$ the sequence of pointed sets

$$[P, W] \leftarrow [Q, W] \leftarrow [R, W]$$

is exact.

**2.3.7 Definition. An unstable exact triangle** is defined to be a sequence $U \to V \to W \to \Sigma_*(U)$ in $\mathbf{CW}^{\mathbf{h}}_*$ which is equivalent to some mapping sequence.

**2.3.8 Proposition** ([Mar83], p.4)**.** *Unstable exact triangles satisfy:*

*(i) they are replete in the appropriate diagram category,*

*(ii)* $0 \to X \xrightarrow{id} X \to 0$ *is an unstable exact triangle,*

*(iii) if* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma_*(X)$ *is an unstable exact triangle then so is*

$$Y \xrightarrow{g} X \xrightarrow{h} \Sigma_*(X) \xrightarrow{-\Sigma_*(f)} \Sigma_*(Y)$$

*(where the minus sign arises from the group structure of* $[\Sigma_*(X), \Sigma_*(Y)]$*),*

*(iv) given a morphism* $f \in [X, Y]$ *there exists an unstable exact triangle* $X \xrightarrow{f} Y \to Z \to \Sigma_*(X)$*,*

*(v) given the following commuting diagram where rows are unstable exact triangles*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] & \Sigma_*(X) \arrow[d, "\Sigma_*(u)"] \\ U \arrow[r] & V \arrow[r] & W \arrow[r] & \Sigma_*(U), \end{tikzcd}$$

*then there exists a fill-in map* $Z \to W$ *(making the diagram commute).*

Another key property of the homotopy category is that the smash product inside $\mathbf{CW}^{\mathbf{h}}_*$ is well-behaved. Precisely we have:

**2.3.9 Proposition** ([Mar83], p.5)**.**

*(a) The smash product is associative:* $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z)$*.*

*(b) There exists a natural isomorphism* $X \wedge Y \cong Y \wedge X$*, i.e. the smash product is commutative.*

*(c) The smash product has unit* $S^0$*:*

$$S^0 \wedge X = X = X \wedge S^0.$$

*(d) There is a natural isomorphism* $e(X, Y) : \Sigma(X) \wedge Y \to \Sigma(X \wedge Y)$.

*(e) Given an unstable exact triangle* $X \to Y \to Z \to \Sigma(X)$ *and any object* $W$ *in* $\mathbf{CW}^{\mathbf{h}}_*$*, then*

$$X \wedge W \to Y \wedge W \to Z \wedge W \to \Sigma(X \wedge W)$$

*is also an unstable exact triangle.*

*(f) The natural map* $\bigvee(X \wedge Y_\alpha) \to X \wedge (\bigvee Y_\alpha)$ *is an isomorphism in* $\mathbf{CW}^{\mathbf{h}}_*$.

The following fundamental result is the basis of stable homotopy theory.

**2.3.10 Theorem** (Freudenthal Suspension Theorem, [Hat02], p.360)**.** *The morphism* $\Sigma : [X, Y] \to [\Sigma(X), \Sigma(Y)]$ *is a bijection if* $\dim X < 2 \cdot \dim Y - 2$.

Essentially this means that, if the condition on the dimension of $X$ is met, then the suspension of $X$ is simply a shifted copy of $X$. Note that it follows that for any finite-dimensional $X$ and $Y$, there is some $N \in \mathbb{N}$ such that

$$\Sigma : [\Sigma^N(X), \Sigma^N(Y)] \to [\Sigma^{N+1}(X), \Sigma^{N+1}(Y)]$$

is a bijection; i.e. the homotopy sets of maps stabilise after sufficiently many suspensions.

## 2.4 The Spanier–Whitehead category, $\mathbf{SW}(\mathbf{CW}_*)$

**2.4.1 Definition. The Spanier–Whitehead category**, $\mathbf{SW}(\mathbf{CW}_*)$, has as objects the pairs $(X, m)$, where $X \in \mathbf{CW}_*$ and $m \in \mathbb{Z}$, and morphisms

$$\mathrm{Hom}_{\mathbf{SW}(\mathbf{CW}_*)}\big((X, m), (Y, n)\big) := \mathrm{colim}_{k \to \infty}[\Sigma^{k+m}_*(X), \Sigma^{k+n}_*(Y)]$$

(where $[P, Q]$ denotes the set of pointed homotopy classes of basepoint-preserving cellular maps).

So in $\mathbf{SW}(\mathbf{CW}_*)$ the morphisms between two objects are defined to be the colimits of the pointed homotopy classes of basepoint-preserving cellular maps between iterated suspensions of the two objects. Freudenthal's Suspension Theorem, 2.3.10, tells us that this colimit is actually attained after a finite number of suspensions:

$$\operatorname{colim}_{k\to\infty}[\Sigma_*^{k+m}(X),\Sigma_*^{k+n}(Y)]\cong[\Sigma_*^{K+m}(X),\Sigma_*^{K+n}(Y)]$$

for sufficiently large $K$.

Thus two objects, $(X,m)$ and $(Y,n)$, in $\mathbf{SW}(\mathbf{CW}_*)$ are isomorphic if and only if $\Sigma_*^{K+m}(X)$ and $\Sigma_*^{K+n}(Y)$ are homotopy equivalent for $K$ sufficiently large. Hence isomorphism in $\mathbf{SW}(\mathbf{CW}_*)$ is called *stable homotopy equivalence.* Composition of morphisms in $\mathbf{SW}(\mathbf{CW}_*)$ is defined by composition of suitably suspended representatives (suspended so that composition is possible).

The relationship between $\mathbf{SW}(\mathbf{CW}_*)$ and the stepping-stone homotopy category, $\mathbf{CW}_*^{\mathbf{h}}$, is formalised by the stabilisation functor

$$\mathrm{Stab}:\mathbf{CW}_*^{\mathbf{h}}\to\mathbf{SW}(\mathbf{CW}_*)$$

given by $X\mapsto(X,0)$. So, in the Spanier–Whitehead category, the finite pointed CW-complex $X$ is identified with the object $(X,0)$. Thus, in $\mathbf{SW}(\mathbf{CW}_*)$, two complexes are thought of as being isomorphic if and only if, after a finite number of suspensions, they become homotopy equivalent.

**2.4.2 Lemma.** *Suppose* $(X,m+1)$ *in* $\mathbf{SW}(\mathbf{CW}_*)$ *where* $X\in\mathbf{CW}_*$ *and* $m\in\mathbb{Z}$. *Then*

$$(X,m+1)\cong(\Sigma_*(X),m)$$

*in* $\mathbf{SW}(\mathbf{CW}_*)$.

*Proof.* Consider two objects $(X,m+1)$ and $(Y,n)$ in $\mathbf{SW}(\mathbf{CW}_*)$. By the definition of the morphism sets in $\mathbf{SW}(\mathbf{CW}_*)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{SW}(\mathbf{CW}_*)}\big((X,m+1),(Y,n)\big)&=\operatorname{colim}_{k\to\infty}[\Sigma_*^{k+m+1}(X),\Sigma_*^{k+n}(Y)]\\&=\mathrm{Hom}_{\mathbf{SW}(\mathbf{CW}_*)}\big((\Sigma_*(X),m),(Y,n)\big)\end{aligned}$$

Hence, by the Yoneda Lemma (details can be found in [Lei14, p.94]), we obtain $(X,m+1)\cong(\Sigma_*(X),m)$. □

A crucial difference between $\mathbf{CW}_*^{\mathbf{h}}$ and $\mathbf{SW}(\mathbf{CW}_*)$ is that the latter allows the existence of desuspensions. In fact $\mathbf{SW}(\mathbf{CW}_*)$ is exactly the category obtained from $\mathbf{CW}_*^{\mathbf{h}}$ by inverting the suspension functor. We have:

- A formal suspension $T:\mathbf{SW}(\mathbf{CW}_*)\to\mathbf{SW}(\mathbf{CW}_*)$ defined by $T(X,m)=(X,m+1)$ which is a functorial automorphism.

- The induced geometric suspension $\Sigma_* : \mathbf{SW}(\mathbf{CW}_*) \to \mathbf{SW}(\mathbf{CW}_*)$ which is given by $\Sigma_*(X, m) = (\Sigma_*(X), m)$.

**2.4.3 Proposition.** *There is a natural isomorphism $T(X, m) \cong \Sigma_*(X, m)$.*

*Proof.* By definition we have $T(X, m) \cong (X, m + 1)$ and by Lemma 2.4.2, we know that $\Sigma_*(X, m) \cong (\Sigma_*(X), m) \cong (X, m + 1)$. □

By taking the colimit, we are not considering sets of pointed homotopy classes of maps from $X$ to $Y$ per se, but instead stabilised versions of these. Since the geometric suspension is invertible, every object in $\mathbf{SW}(\mathbf{CW}_*)$ can be described as a double suspension. Thus the stabilised versions of the sets of pointed homotopy classes are abelian groups:

**2.4.4 Proposition.** *In the Spanier–Whitehead category the morphism sets*

$$\mathrm{Hom}_{\mathbf{SW}(\mathbf{CW}_*)}\big((X, m), (Y, n)\big)$$

*are abelian groups.*

**2.4.5 Lemma** ([Mar83, p.8])**.** *The Spanier–Whitehead category,* $\mathbf{SW}(\mathbf{CW}_*)$, *is an additive category.*

*Proof.* Clearly the zero object of $\mathbf{SW}(\mathbf{CW}_*)$ is the basepoint $*$. The category inherits an additive structure as $\mathrm{Hom}_{\mathbf{SW}(\mathbf{CW}_*)}\big(A, B\big)$ is the natural colimit of abelian groups. The coproduct in $\mathbf{SW}(\mathbf{CW}_*)$ is

$$(X, m) \amalg (Y, n) := (\Sigma_*^{l-n}(X) \vee \Sigma_*^{l-m}(Y), m + n - l)$$

where $l \in \mathbb{Z}$ is such that $l - m, l - n \geq 0$ for each pair $m, n \in \mathbb{Z}$. Since the morphism sets in $\mathbf{SW}(\mathbf{CW}_*)$ are abelian groups, this is also a product. Thus the wedge sum gives us a sum in $\mathbf{SW}(\mathbf{CW}_*)$.

□

**2.4.6 Definition.** The **collection of distinguished triangles, $\boldsymbol{\Delta}$**, in $\mathbf{SW}(\mathbf{CW}_*)$ consists of mapping sequences $X \xrightarrow{f} Y \to \mathrm{C}_*(f) \to \Sigma_*(X)$ together with their suspensions (and de-suspensions). Hence any sequence in $\mathbf{SW}(\mathbf{CW}_*)$ of the form $A \to B \to C \to \Sigma_*(A)$ that is equivalent to a mapping sequence $X \xrightarrow{f} Y \to \mathrm{C}_*(f) \to \Sigma_*(X)$ is in $\Delta$.

**2.4.7 Theorem** ([Mar83, p.8])**.** *The Spanier–Whitehead category,* $\mathbf{SW}(\mathbf{CW}_*)$, *satisfies axioms* **TR.1** - **TR.5** *of a triangulated category.*

*Proof.* We verify that the triple $(\mathbf{SW}(\mathbf{CW}_*), \Sigma_*, \Delta)$ satisfies axioms **TR.1** - **TR.5** of a triangulated category.

**TR.1** $\Delta$ is replete: this follows from the definition of $\Delta$ in $\mathbf{SW}(\mathbf{CW}_*)$.

**TR.2** $0 \to A \xrightarrow{id} A \to 0$ is in $\Delta$: this follows from $(ii)$ Proposition 2.3.8.

**TR.3** Given a morphism $f \in \mathrm{Hom}_{\mathbf{SW}(\mathbf{CW}_*)}\big((A,m),(B,n)\big)$, there exists a triangle $A \xrightarrow{f} B \to C \to \Sigma_*(A)$ : this follows from the definition of a mapping sequence and triangles in $\mathbf{SW}(\mathbf{CW}_*)$.

**TR.4** $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma_*(A)$ is in $\Delta$ if and only if $B \xrightarrow{g} C \xrightarrow{h} \Sigma_*(A) \xrightarrow{-\Sigma_*(f)} \Sigma_*(B)$ is in $\Delta$: this follows from $(iii)$ of Proposition 2.3.8.

**TR.5** Given the following commuting diagram where rows are exact triangles

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "h"] & \Sigma_*(A) \arrow[d, "\Sigma_*(f)"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma_*(A'),
\end{tikzcd}$$

then there exists a fill-in map $h : C \to C'$ making the diagram commute: it suffices to apply axiom $(v)$ of Proposition 2.3.8.

$\square$

**2.4.8 Remark.** In fact $\mathbf{SW}(\mathbf{CW}_*)$ also satisfies **TR.6** (although Margolis doesn't give a proof of this in [Mar83]). We will provide a proof in a more general context in Theorem 4.2.6.

# Chapter 3

# An axiomatic approach to Spanier–Whitehead categories

In this chapter, we advance an axiomatic approach to constructing the Spanier-Whitehead category given some ambient category with some specific properties. We then prove that axioms **TR.1** to **TR.6** hold, and hence that the corresponding Spanier–Whitehead category is triangulated. This will produce an efficient method for determining whether a given category gives rise to a Spanier–Whitehead category.

In particular, this approach will allow us to verify that $\mathbf{CW}_*$ satisfies these ambient category conditions. Thus, making use of this axiomatic method, we will be able to put forward a neat proof that $\mathbf{SW}(\mathbf{CW}_*)$ is triangulated which does include the octahedral axiom (we recall that Margolis' verification that $\mathbf{SW}(\mathbf{CW}_*)$ is a triangulated category, in [Mar83], does not include the octahedral axiom, **TR.6**). However, our primary motivation for this axiomatic approach via an ambient category is to apply it to the category **Def** of compact definable spaces in some fixed o-minimal expansion of $\mathbb{R}$. This will allow us to construct Spanier–Whitehead categories $\mathbf{SW}(\mathbf{Def}_Z)$ for each slice-coslice category of **Def**. In fact this is a slight simplification. The category **Def** does not quite satisfy all the required properties, but we show it satisfies enough for the construction to go through. This is discussed in detail in Chapter 4.

The structure of Chapter 3 is as follows. In Section 3.1 we present conditions on a category $\mathbf{C}$ that suffice to construct a Spanier–Whitehead category $\mathbf{SW}(\mathbf{C}_Z)$ for each slice-coslice category $\mathbf{C}_Z := Z\backslash\mathbf{C}/Z$ with $Z \in \mathbf{C}$. We then examine the slice-coslice category $\mathbf{C}_Z$ in detail in Section 3.2. The whole of Section 3.3 is dedicated to coexact (Puppe) mapping sequences in $\mathbf{C}_Z$ which are ubiquitous in the Spanier–Whitehead category. In Section 3.4, we compare two slice-coslice categories relative to two different fixed base-objects given a morphism between the two base-objects. We prove that there exist adjoint functors between the two slice-coslice categories which preserve sufficient structure. The left adjoint is given by a pushout, and the right adjoint a pullback. Section 3.5 consists of the definition of the Spanier–Whitehead category of the slice-coslice category,

$\mathbf{SW}(\mathbf{C}_Z)$, and a proof that it is a triangulated category. Finally Section 3.6 contains a proof that the adjunction in Section 3.4 descends to an adjunction between the corresponding Spanier–Whitehead categories where both the aforementioned functors descend to triangulated functors between the Spanier–Whitehead categories and the pullback functor is monoidal. We refer to this as the base-change functoriality. We also briefly discuss the change-of-ambience functor: given a functor between two categories satisfying the ambient category conditions, preserving sufficient structure, there are induced triangulated functors between the Spanier–Whitehead categories of the corresponding slice-coslice categories.

## 3.1 An ambient category C

We start by laying out conditions on an ambient category $\mathbf{C}$ that suffice in order to be able to construct a Spanier–Whitehead category $\mathbf{SW}(\mathbf{C}_Z)$ for each slice-coslice category $\mathbf{C}_Z := Z\backslash\mathbf{C}/Z$ with $Z \in \mathbf{C}$. Suppose that $\mathbf{C}$ is a category with the following properties $\mathbf{P.1} - \mathbf{P.5}$.

**P.1 C has finite limits and finite colimits;**

**P.2 Fibre products preserve finite colimits in C;**

**P.3 C has a chosen interval object, $I \in \mathbf{C}$, such that a factorisation of the fold morphism $S^0 \to *$ via the morphism $S^0 \to I$ exists:**

$$\begin{tikzcd} S^0 \arrow[rr, "id_* \vee_\emptyset id_*"] \arrow[dr, dashed, "c_*"'] & & * \\ & I \arrow[ur, "\pi_I"'] & \end{tikzcd}$$

**where the 0-sphere $S^0$ in C is defined to be $S^0 := * \vee_\emptyset *$;**

**P.4 The morphism $c_* : S^0 \to I$ is a cofibration in C;**

**P.5 The transposition morphism $\tau : S^0 \to S^0$ in C extends to the commutative square**

$$\begin{tikzcd} S^0 \arrow[r, "c_*"] \arrow[d, "\tau"] & I \arrow[d, dashed, "\tau"] \\ S^0 \arrow[r, "c_*"] & I, \end{tikzcd}$$

**so that $\tau^2 = id$ in C.**

We now unpack these properties, and explain how they are used.

By **P.1** the category **C** has an initial object $\emptyset \xrightarrow{\iota_X} X$ and a terminal object $X \xrightarrow{\pi_X} *$.

**3.1.1 Definition.** Given two morphisms $f : A \to X$ and $g : A \to Y$, the **pushout** of $f$ and $g$ in **C** consists of an object $X \vee_A Y$ together with two morphisms $k_1 : X \to X \vee_A Y$ and $k_2 : Y \to X \vee_A Y$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "k_1"] \\ Y \arrow[r, "k_2"'] \arrow[ur, phantom, "\ulcorner"] & X \vee_A Y, \end{tikzcd}$$

and such that $(X \vee_A Y, k_1, k_2)$ has the universal property with respect to this diagram.

**3.1.2 Remark.** Of course the standard properties of pushouts hold in **C**. In particular:

(i) pushouts are insensitive to ordering: whenever a pushout $X \vee_A Y$ exists, $Y \vee_A X$ exists and there is a natural isomorphism $\tau : X \vee_A Y \longrightarrow Y \vee_A X$;

(ii) the pasting law for pushouts holds: given a commuting diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[d] \\ W' \arrow[r] & X' \arrow[r] & Y', \end{tikzcd}$$

if the left hand square is a pushout, then the outer rectangle is a pushout if and only if the right hand square is a pushout. Notably, given

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[r, "h"] \arrow[d] & D \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A \vee_C B \arrow[r] \arrow[ur, phantom, "\ulcorner"] & (A \vee_C B) \vee_B D, \end{tikzcd}$$

the outer rectangle is also a pushout. Hence there exists a natural isomorphism

$$(A \vee_C B) \vee_B D \cong A \vee_C D.$$

The coproduct of $X$ and $Y$ in **C** is obtained by setting $A$ to be the initial object, i.e. $A = \emptyset$, in the above definition:

$$\begin{tikzcd} \emptyset \arrow[r, "\iota_X"] \arrow[d, "\iota_Y"'] \arrow[dr, phantom, "\ulcorner", very near end] & X \arrow[d] \\ Y \arrow[r] & X \vee_{\emptyset} Y. \end{tikzcd}$$

We write the pushout of $* \xleftarrow{\pi_X} X \xrightarrow{f} Y$ in $\mathbf{C}$ as $Y/X$ , i.e.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\ulcorner", very near end] & Y \arrow[d] \\ * \arrow[r] & Y/X. \end{tikzcd}$$

**3.1.3 Definition.** Given two morphisms $f : X \to B$ and $g : Y \to B$, the **pullback** (also known as the **fibre product**) of $f$ and $g$ in $\mathbf{C}$ consists of an object $X \times_B Y$ together with two morphisms $l_1 : X \times_B Y \to X$ and $l_2 : X \times_B Y \to Y$ such that the following diagram commutes

$$\begin{tikzcd} X \times_B Y \arrow[r, "l_1"] \arrow[d, "l_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & B, \end{tikzcd}$$

and such that $(X \times_B Y, l_1, l_2)$ has the universal property with respect to this diagram.

**3.1.4 Remark.** The standard properties of fibre products also hold in $\mathbf{C}$. In particular:

(i) fibre products are insensitive to ordering: whenever $A \times_C B$ exists, $B \times_C A$ exists and there is a natural isomorphism $A \times_C B \cong B \times_C A$;

(ii) there exists a natural isomorphism $(A \times_C B) \times_B D \cong A \times_C D$.

(iii) the pasting law for fibre products holds.

The product of $X$ and $Y$ in $\mathbf{C}$ is obtained by setting $B$ to be the final object, i.e. $B = *$, in the above definition:

$$\begin{tikzcd} X \times Y \arrow[r, "l_1"] \arrow[d, "l_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\pi_X"] \\ Y \arrow[r, "\pi_Y"'] & *, \end{tikzcd}$$

By **P.2**, fibre products distribute over pushouts in $\mathbf{C}$, i.e.

$$W \times_B (X \vee_A Y) \cong (W \times_B X) \underset{W \times_B A}{\vee} (W \times_B Y). \tag{3.1.1}$$

**P.3** provides **C** with a suitable interval object. We denote the composite morphisms

$$* \xrightarrow{\iota_0} S^0 \xrightarrow{c_*} I \text{ and } * \xrightarrow{\iota_1} S^0 \xrightarrow{c_*} I$$

by $\iota_0 : * \to I$ and $\iota_1 : * \to I$ respectively. We write $0 := \iota_0(*) \in I$ and $1 := \iota_1(*) \in I$. The notions of product, pushout and interval object are required to define the following constructions in **C**.

**3.1.5 Definition.** Consider any object $X$ in **C**.

1. The **cylinder** on $X$ is defined to be $\mathrm{Cyl}(X) := X \times I \in \mathbf{C}$.

2. The **cone** on $X$, denoted by $C(X)$, is defined via the pushout

$$\begin{tikzcd} X \arrow[r, "\pi_X"] \arrow[d, "id_X \times \iota_0"'] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\ \mathrm{Cyl}(X) \arrow[r] & C(X). \end{tikzcd}$$

**3.1.6 Definition.** Consider any morphism $f : X \to Y$ in **C**.

1. The **mapping cylinder** of $f$, denoted by $\mathrm{Cyl}(f)$ in **C**, is defined via the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X \times \iota_1"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ \mathrm{Cyl}(X) \arrow[r] & \mathrm{Cyl}(f). \end{tikzcd}$$

2. The **mapping cone** of $f$, denoted by $C(f)$ in **C**, is defined via the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id_X \times \iota_1"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ C(X) \arrow[r] & C(f). \end{tikzcd}$$

**3.1.7 Definition.** The **suspension** of $X$, denoted by $S(X)$ in **C**, is defined to be the mapping cone of $\pi_X : X \to *$, i.e.

$$\begin{tikzcd} X \arrow[r, "\pi_X"] \arrow[d, "id_X \times \iota_1"'] \arrow[dr, phantom, "\ulcorner"] & * \arrow[d] \\ C(X) \arrow[r] & S(X). \end{tikzcd}$$

So $S(X) = C(\pi_X)$.

Consider the morphism $c_* : S^0 \to I$ in **P.3**. We have $X \times S^0 \cong X \vee_\emptyset X$ by **P.2**. We write

$$\bar{c_*} := id_X \times c_* : X \vee_\emptyset X \longrightarrow X \times I.$$

In order to define homotopy in **C** the first three properties, **P.1**, **P.2** and **P.3**, are used.

**3.1.8 Definition.** Given two morphisms $f, g : X \to Y$ in **C**, a **homotopy** from $f$ to $g$ is a morphism $h : X \times I \to Y$ such that the following diagram commutes

$$\begin{tikzcd} X \vee_\emptyset X \arrow[rr, "f \vee_\emptyset g"] \arrow[dr, "\bar{c_*}"'] & & Y. \\ & X \times I \arrow[ur, dashed, "h"'] & \end{tikzcd} \tag{3.1.2}$$

The morphisms $f$ and $g$ are **homotopic**, $f \simeq g$, if such a homotopy exists.

**3.1.9 Definition.** A morphism $f : X \to Y$ in **C** is a **homotopy equivalence** if there exists a morphism $g : Y \to X$ (a homotopy inverse) such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. The objects $X$ and $Y$ in **C** are then said to be **homotopy equivalent**, $X \simeq Y$.

The morphism in the following lemma will be required in various definitions and proofs.

**3.1.10 Lemma.** *Given $f : A \to B$ in* **C** *there exists a morphism*

$$(A \times I) \vee_{A \times 0} (B \times 0) = \mathrm{Cyl}(f) \xrightarrow{\quad i \quad} B \times I$$

*restricting to $f \times id_I$ on $A \times I$ and to $id \times \iota_0$ on $B \times 0$.*

*Proof.* Given $f : A \to B$ in **C**, the morphism $i : \mathrm{Cyl}(f) \longrightarrow B \times I$ exists uniquely by the universal property of the pushout diagram

$$\begin{tikzcd} A \times 0 \arrow[r, "f \times id_0"] \arrow[d, "id_A \times \iota_0"'] & B \times 0 \arrow[d] \arrow[ddr, bend left, "id \times i_0"] & \\ A \times I \arrow[r] \arrow[drr, bend right, "f \times id_I"'] \arrow[ur, phantom, "\ulcorner"] & (A \times I) \vee_{A \times 0} (B \times 0) \arrow[dr, dashed, "\exists ! i"] & \\ & & B \times I. \end{tikzcd}$$

□

**3.1.11 Definition.** A morphism $f : A \to B$ in **C** is a **cofibration** if it has the homotopy extension property, i.e. if, for any morphism $\mathrm{Cyl}(f) \to X$, there exists a morphism $G$ which completes the following diagram (not necessarily uniquely) to a commutative diagram:

$$\begin{tikzcd} \mathrm{Cyl}(f) \arrow[r, "i"] \arrow[d] & B \times I \arrow[dl, dashed, "\exists G"] \\ X & \end{tikzcd}$$

where $\mathrm{Cyl}(f) = (A \times I) \vee_{A\times 0} (B \times 0)$. In other words, a morphism $f : A \to B$ is a cofibration if, whenever there exists a morphism $j : B \times 0 \to X$ and a homotopy $J : A \times I \to X$ such that $J|_{A\times 0} = j \circ (f \times id_0)$, then the homotopy can be extended to $G : B \times I \to X$ so that $G \circ (f \times id_I) = J$ and $G|_{B\times 0} = j$.

**3.1.12 Lemma.** *A morphism $f : A \to B$ is a cofibration in* **C** *if and only if* $\mathrm{Cyl}(f)$ *is a retract of $B \times I$, i.e. there exist morphisms*

$$\mathrm{Cyl}(f) \xrightarrow{\quad i \quad} B \times I \xrightarrow{\quad r \quad} \mathrm{Cyl}(f)$$

*such that $r \circ i = id$, so that $r$ is a retraction of $B \times I$ onto* $\mathrm{Cyl}(f) = (A \times I) \vee_{A\times 0} (B \times 0)$.

*Proof.* Assume $f : A \to B$ is a cofibration in **C**. Then, considering

$$\begin{tikzcd} \mathrm{Cyl}(f) \arrow[r, "i"] \arrow[d, "id"'] & B \times I \arrow[dl, dashed, "\exists r"] \\ \mathrm{Cyl}(f) & \end{tikzcd}$$

we obtain a retraction $r : B \times I \longrightarrow \mathrm{Cyl}(f)$. The opposite direction is obvious: given a retraction $B \times I \longrightarrow \mathrm{Cyl}(f)$, then clearly $f : A \to B$ is a cofibration. $\square$

**3.1.13 Lemma** (Properties of cofibrations in **C**)**.**

*(i) Cofibrations are preserved by product in* **C**.

*(ii) Cofibrations are closed under composition in* **C**.

*(iii) Cofibrations are closed under pushout in* **C**.

*Proof.*

(i) Suppose $f : A \to B$ is a cofibration in **C**. Then, by Lemma 3.1.12 there exists a retraction

$$r : B \times I \longrightarrow (A \times I) \vee_{A\times 0} (B \times 0).$$

Let $C$ be any object in $\mathbf{C}$. Then, since product preserves pushouts in $\mathbf{C}$ by **P.2**, the following is also a retraction

$$r \times id_C : (B \times I) \times C \longrightarrow (A \times C \times I) \vee_{A\times C\times 0} (B \times C \times 0).$$

Hence, again by Lemma 3.1.12, the morphism $f \times id_C : A \times C \to B \times C$ is also a cofibration in $\mathbf{C}$.

(ii) Suppose $f : A \to B$ and $g : B \to C$ are cofibrations in $\mathbf{C}$. For simplicity, we use the notation $\mathrm{Cyl}(f) = (A\times I)\vee_f B$ and $\mathrm{Cyl}(g) = (B\times I)\vee_g C$. Consider the composite morphism $g \circ f : A \to C$. We write $\mathrm{Cyl}(gf) = (A \times I) \vee_{gf} C$. By writing out the definitions of these mapping cylinders as pushouts, we obtain the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner"] & A \times I \arrow[d] & \\
B \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Cyl}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \times I \arrow[d] \\
C \arrow[r] & \mathrm{Cyl}(gf) \arrow[r] & \mathrm{Cyl}(g).
\end{tikzcd}$$

Hence we have a pushout square

$$\begin{tikzcd}
\mathrm{Cyl}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \times I \arrow[d] \\
\mathrm{Cyl}(gf) \arrow[r] & \mathrm{Cyl}(g).
\end{tikzcd}$$

By the definition of a cofibration, given a morphism $\mathrm{Cyl}(f) \to X$, there exists a morphism $F : B \times I \to X$ making

$$\begin{tikzcd}
\mathrm{Cyl}(f) \arrow[r] \arrow[d] & B \times I \arrow[dl, dashed, "\exists F"] \\
X &
\end{tikzcd}$$

commute. Similarly, given a morphism $\mathrm{Cyl}(g) \to X$, there exists $G : C \times I \to X$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Cyl}(g) \arrow[r] \arrow[d] & C \times I \arrow[dl, dashed, "\exists G"] \\
X &
\end{tikzcd}$$

Suppose there exists a morphism $\mathrm{Cyl}(gf) \to X$. Then to see that the composite $g \circ f$ is a cofibration, consider the following diagram

$$
\begin{tikzcd}
\mathrm{Cyl}(f) \arrow[r] \arrow[d] & B \times I \arrow[d] \arrow[ddl, dashed, "F"'] & \\
\mathrm{Cyl}(gf) \arrow[r] \arrow[d] & \mathrm{Cyl}(g) \arrow[r] \arrow[dl, dotted, "\exists!"] \arrow[ul, phantom, "\ulcorner", very near start] & C \times I, \arrow[dll, dashed, bend left, "G"'] \\
X & &
\end{tikzcd}
$$

where, by the universal property of pushouts, there exists a unique morphism $\mathrm{Cyl}(g) \to X$.

(iii) Suppose $f : A \to B$ is a cofibration. Consider the pushout of $C \leftarrow A \xrightarrow{f} B$ for some $C$ in $\mathbf{C}$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
C \arrow[r, "g"'] & D. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

To prove that $g : C \to D$ is also a cofibration, we first patchwork together various pushout squares to show that the following square is a pushout:

$$
\begin{tikzcd}
\mathrm{Cyl}(f) \arrow[r] \arrow[d] & B \times I \arrow[d] \\
\mathrm{Cyl}(g) \arrow[r] & D \times I.
\end{tikzcd}
$$

By Remark 3.1.2, (ii), the outer rectangle in the following diagram is a pushout:

$$
\begin{tikzcd}
A \times 1 \arrow[r] \arrow[d] & B \arrow[d] \\
C \times 1 \arrow[r] \arrow[d] & D \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] \\
C \times I \arrow[r] & \mathrm{Cyl}(g). \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

Now consider

$$
\begin{tikzcd}
A \times 1 \arrow[r] \arrow[d] & B \arrow[d] \\
A \times I \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \mathrm{Cyl}(f) \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] \\
C \times I \arrow[r] & \mathrm{Cyl}(g)
\end{tikzcd}
$$

where the top square is the pushout defining $\mathrm{Cyl}(f)$ and the outer rectangle is a pushout by above. Thus, by the pasting law for pushouts, the bottom square, 1, is also a pushout. By **P.2** product preserves pushouts, the following diagram is a pushout in $\mathbf{C}$

$$\begin{tikzcd} A \times I \arrow[r, "f\times id_I"] \arrow[d] & B \times I \arrow[d] \\ C \times I \arrow[r, "g\times id_I"'] \arrow[ur, phantom, "\ulcorner"] & D \times I. \end{tikzcd}$$

Then, considering

$$\begin{tikzcd} A \times I \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \mathrm{Cyl}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & B \times I \arrow[d] \\ C \times I \arrow[r] & \mathrm{Cyl}(g) \arrow[r] & D \times I, \end{tikzcd}$$

by the pasting law square 2 is a pushout, as required. Since $f$ is a cofibration, given a morphism $\mathrm{Cyl}(f) \to X$, there exists a homotopy $H : B \times I \to X$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Cyl}(f) \arrow[r] \arrow[d] & B \times I \arrow[dl, dashed, "\exists H"] \\ X. & \end{tikzcd}$$

Therefore, given a morphism $\mathrm{Cyl}(g) \to X$, by the universal property of pushout 2, there exists a unique morphism $D \times I \to X$ as below:

$$\begin{tikzcd} \mathrm{Cyl}(f) \arrow[r] \arrow[d] & B \times I \arrow[d] \arrow[ddl, bend right, "H"'] \\ \mathrm{Cyl}(g) \arrow[r] \arrow[d] \arrow[d, bend left] & D \times I \arrow[dl, dashed, "\exists!"] \\ X. & \end{tikzcd}$$

Hence $g : C \to D$ is a cofibration.

$\square$

**3.1.14 Lemma.** *For any $X \in \mathbf{C}$, the morphism $\emptyset \xrightarrow{\iota_X} X$ is a cofibration in $\mathbf{C}$.*

*Proof.* To prove that $\iota_X : \emptyset \to X$ is a cofibration, we need to prove there exists a retraction $X \times I \longrightarrow \mathrm{Cyl}(\iota_X)$. Since $\mathrm{Cyl}(\iota_X) = (\emptyset \times I) \vee_{\iota_X} X \cong \emptyset \vee_{\iota_X} X \cong X$, we have the composite

$$X \xrightarrow{\quad id_X \times \iota_0 \quad} X \times I \xrightarrow{\quad id_X \times \pi_I \quad} X$$

which is the identity. Thus $X \times I \longrightarrow X \cong \mathrm{Cyl}(\iota_X)$ is a retraction, as required. $\Box$

**3.1.15 Remark.** It follows from Lemma 3.1.14 that every object in the ambient category $\mathbf{C}$ is 'cofibrant'.

The definitions and lemmas above depend only on **P.1**, **P.2** and **P.3**. We now present some further properties of homotopies in $\mathbf{C}$ which, in addition, require the morphism $c_* : S^0 \to I$ to be cofibration, **P.4**, and the existence of a transposition map $\tau : I \to I$ in $\mathbf{C}$ as in **P.5**.

**3.1.16 Remark.** Consider the definition of the mapping cylinder of $c_*$ as below:

$$\begin{tikzcd} S^0 \arrow[r, "c_*"] \arrow[d, "id_{S^0} \times \iota_1"'] \arrow[dr, phantom, "\ulcorner"] & I \arrow[d] \\ S^0 \times I \arrow[r] & \mathrm{Cyl}(c_*) \end{tikzcd}$$

where $S^0 \times I \cong ((* \vee_{\emptyset} *) \times I) \cong (I \vee_{\emptyset} I)$. It is then not difficult to see that the mapping cylinder of $c_*$ consists of an interval $I$ which has, at either "end", copies of $I$ attached. In particular

$$\mathrm{Cyl}(c_*) \cong I \underset{1=0}{\vee} I \underset{1=1}{\vee} I$$

where $I \underset{1=0}{\vee} I$ denotes the pushout of

$$I \xleftarrow{\iota_1} * \xrightarrow{\iota_0} I$$

and $I \underset{1=1}{\vee} I$ to denotes the pushout of

$$I \xleftarrow{\iota_1} * \xrightarrow{\iota_1} I.$$

This description of the mapping cylinder of $c_* : S^0 \to I$ will be a useful in proving the following lemmas.

**3.1.17 Lemma.** *The interval object $I$ is contractible in* $\mathbf{C}$, *i.e.* $I \simeq *$.

*Proof.* We need to prove that the morphisms $\iota_0 : * \to I$ and $\pi_I : I \to *$ are homotopy equivalences. Clearly $\pi_I \circ \iota_0 = id_*$. We denote the constant morphism $\iota_0 \circ \pi_I : I \to * \to I$ by $e_0$. To show that $e_0 \simeq id_I$, we need to construct a homotopy $H : I \times I \to I$ from the identity on $I$ to the constant morphism $e_0 : I \to I$. We use the fact that the morphism $c_* : S^0 \to I$ is a cofibration (by **P.4**) to construct such a homotopy. Consider the description of the mapping cylinder of $c_*$ as in Remark 3.1.16:

$$\mathrm{Cyl}(c_*) \cong I \underset{1=0}{\vee} I \underset{1=1}{\vee} I.$$

By Lemma 3.1.10 there exists a unique morphism $i : \mathrm{Cyl}(c_*) \longrightarrow I \times I$. Take the morphism

$$id \underset{1=0}{\vee} \tau \underset{1=1}{\vee} e_0 : I \underset{1=0}{\vee} I \underset{1=1}{\vee} I \longrightarrow I$$

(where $\tau$ is the transposition map on $I$ as in **P.5**). Then, since $c_* : S^0 \to I$ is a cofibration by **P.4**, we can construct a homotopy $G : I \times I \to I$ from the identity on $I$ to the constant morphism $e_0$ as required:

$$\begin{tikzcd} \mathrm{Cyl}(c_*) \arrow[r, "i"] \arrow[d, "id \underset{1=0}{\vee} \tau \underset{1=1}{\vee} e_0"'] & I \times I \arrow[dl, dashed, "\exists G"] \\ I. & \end{tikzcd}$$

□

**3.1.18 Definition.** The **double interval** object in $\mathbf{C}$ is $I' = I \underset{1=0}{\vee} I$, i.e.

$$\begin{tikzcd} * \arrow[r, "\iota_0"] \arrow[d, "\iota_1"'] & I \arrow[d, "\beta"] \\ I \arrow[r, "\alpha"'] & I' \end{tikzcd} \tag{3.1.3}$$

where $\beta(0) = \alpha(1) = 1 \in I'$. Write $0 = \alpha(0) \in I'$ and $2 = \beta(1) \in I'$.

**3.1.19 Lemma.** *There exists a morphism $\delta : I \to I'$ such that $\delta(0) = 0$ and $\delta(1) = 2$ and a homotopy $\eta : I \times I \to I'$ from $\delta$ to $\alpha$ (the inclusion of the first summand) and relative to* 0.

*Proof.* This relies on the fact that $c_* : S^0 \to I$ is a cofibration, **P.4**. By Remark 3.1.16 we can consider

$$\mathrm{Cyl}(c_*) \cong I \underset{1=0}{\vee} I \underset{1=1}{\vee} I.$$

Firstly we construct $\delta : I \to I'$ where $I' = I \underset{1=0}{\vee} I$. To do this consider the morphism $i : \mathrm{Cyl}(c_*) \to I \times I$ (by Lemma 3.1.10) and

$$u_0 \underset{1=0}{\vee} \alpha \underset{1=1}{\vee} \beta\tau : I \underset{1=0}{\vee} I \underset{1=1}{\vee} I \longrightarrow I \underset{1=0}{\vee} I$$

where $u_0 : I \to I$ is the constant morphism, $\tau : I \to I$ is the transposition as in **P.5**, and $\alpha$ and $\beta$ are as in Definition 3.1.18. Then by **P.4**, there is a homotopy $G : I \times I \to I'$ which completes the diagram

$$\begin{tikzcd} \mathrm{Cyl}(c_*) \arrow[r, "i"] \arrow[d, "u_0 \underset{1=0}{\vee} \alpha \underset{1=1}{\vee} \beta\tau"'] & I \times I \arrow[dl, dashed, "\exists G"] \\ I'. & \end{tikzcd}$$

Considering the diagonal morphism $\varDelta : I \to I \times I$, we can then define $\delta$ to be the composite

$$\delta = G \circ \varDelta : I \longrightarrow I \times I \longrightarrow I'.$$

To construct the homotopy $\eta : I \times I \to I'$ from $\delta$ to $\alpha$ consider

$$\alpha\tau \underset{1=0}{\vee} u_0 \underset{1=1}{\vee} \delta\tau : I \underset{1=0}{\vee} I \underset{1=1}{\vee} I \longrightarrow I \underset{1=0}{\vee} I.$$

Then, again by **P.4**, the required homotopy $\eta : I \times I \to I'$ from $\delta$ to $\alpha$ exists:

$$\begin{tikzcd} \mathrm{Cyl}(c_*) \arrow[r, "i"] \arrow[d, "\alpha\tau \underset{1=0}{\vee} u_0 \underset{1=1}{\vee} \delta\tau"'] & I \times I \arrow[dl, dashed, "\exists\eta"] \\ I'. & \end{tikzcd}$$

□

**3.1.20 Lemma.** *Homotopy is an equivalence relation in* **C**.

*Proof.* Reflexivity: consider the diagram obtained by taking the product with $X$ of the diagram in **P.4**:

$$\begin{tikzcd} X \vee_\emptyset X \arrow[rr, "id_X \vee_\emptyset id_X"] \arrow[dr, "c_X"'] & & X. \\ & X \times I \arrow[ur, "id_X \times \pi_I"'] & \end{tikzcd} \tag{3.1.4}$$

Then, given $f : X \to Y$ in **C**, we have

$$\begin{tikzcd} X \vee_\emptyset X \arrow[rr, "id_X \vee_\emptyset id_X"] \arrow[dr, "c_X"'] & & X \arrow[r, "f"] & Y, \\ & X \times I \arrow[ur, "id_X \times \pi_I"'] \arrow[urr, bend right, "h"'] & & \end{tikzcd}$$

which gives a homotopy $h : X \times I \to Y$ from $f$ to $f$.

To prove symmetry: let $h$ be a homotopy from $f : X \to Y$ to $g : X \to Y$, i.e. fitting in to a commutative diagram

$$\begin{tikzcd} X \vee_\emptyset X \arrow[rr, "f \vee_\emptyset g"] \arrow[dr, "c_X"'] & & Y. \\ & X \times I \arrow[ur, "h"'] & \end{tikzcd}$$

By Remark 3.1.2, (i), there exists a transposition morphism $X \vee_\emptyset X \xrightarrow{\tau} X \vee_\emptyset X$. Then precomposing this with the commuting diagram above, we obtain

$$\begin{tikzcd} \vee_\emptyset X \arrow[r, "\tau"] \arrow[rrr, bend left, "g\vee_\emptyset f"] & X \vee_\emptyset X \arrow[rr, "f\vee_\emptyset g"] \arrow[dr, "c_X"'] & & Y. \\ & & X \times I \arrow[ur, "h"'] & \end{tikzcd}$$

This gives the required homotopy from $g$ to $f$.

To prove transitivity: Let $\epsilon$ be a homotopy between maps $f, g : X \to Y$, i.e. there is a commutative diagram

$$\begin{tikzcd} \vee_\emptyset X \arrow[rr, "f\vee_\emptyset g"] \arrow[dr, "c_X"'] & & Y, \\ & X \times I \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

and $\zeta$ a homotopy between maps $g, h : X \to Y$, i.e. there is a commutative diagram

$$\begin{tikzcd} X \vee_\emptyset X \arrow[rr, "g\vee_\emptyset h"] \arrow[dr, "c_X"'] & & Y. \\ & X \times I \arrow[ur, "\zeta"'] & \end{tikzcd}$$

In order to glue homotopies $\epsilon$ and $\zeta$ together, consider the following pushout obtained from diagram (3.1.3) by taking the product with $X$ (by **P.2**):

$$\begin{tikzcd} X \arrow[r, "id_X \times i_0"] \arrow[d, "id_X \times i_1"'] & X \times I \arrow[d, "id_X \times \beta"] \\ X \times I \arrow[r, "id_X \times \alpha"'] \arrow[ur, phantom, "\ulcorner"] & X \times I'. \end{tikzcd}$$

From the homotopies $\epsilon : X \times I \to Y$ between $f$ and $g$, and $\zeta : X \times I \to Y$ between $g$ and $h$, and the universal property of the above pushout, there exists a unique map $\gamma$ as below:

$$\begin{tikzcd} X \arrow[r, "id_X \times i_0"] \arrow[d, "id_X \times i_1"'] & X \times I \arrow[d, "id_X \times \beta"] \arrow[ddr, bend left, "\zeta"] & \\ X \times I \arrow[r, "id_X \times \alpha"'] \arrow[ur, phantom, "\ulcorner"] \arrow[drr, bend right, "\epsilon"'] & X \times I' \arrow[dr, dashed, "\exists!\gamma"] & \\ & & Y. \end{tikzcd}$$

Thus we obtain a homotopy $\gamma : X \times I' \to Y$ between $f$ and $h$, indexed by $I'$. Since there exists a suitable map $\delta : I \to I'$ (by Lemma 3.1.19), a family of maps indexed by $I'$ can be replaced by a family indexed by $I$. Hence $\gamma : X \times I \to Y$ represents the required homotopy between maps $f$ and $h$. $\square$

**3.1.21 Definition.** The **set of homotopy classes of maps from** $X$ **to** $Y$ in $\mathbf{C}$ is defined to be

$$[X, Y] = \mathbf{C}(X, Y)/\sim$$

where $\sim$ is the homotopy equivalence relation on the set of morphisms from $X$ to $Y$, i.e.

$$[X, Y] = \{[f]_\sim \mid f \in \mathbf{C}(X, Y)\}$$

with $[f]_\sim = \{g \in \mathbf{C}(X, Y) \mid g \sim f\}$.

**3.1.22 Remark.** We note that **P.5** is included for convenience, and that it may be possible to derive it from **P.1**, **P.2**, **P.3** and **P.4**.

## 3.2 The slice-coslice category $\mathbf{C}_Z$

Let $Z \in \mathrm{Ob}(\mathbf{C})$ be some fixed object in the ambient category $\mathbf{C}$. Consider the **slice-coslice category** of $\mathbf{C}$, denoted $\mathbf{C}_Z := \mathbf{Z}\backslash(\mathbf{C}/\mathbf{Z})$. Objects of $\mathbf{C}_Z$ are of the form $Z \xrightarrow{i_X} \mathbf{X} \xrightarrow{p_X} Z$ where $p_x i_x = id_Z$. A morphism in $\mathbf{C}_Z$ from $Z \xrightarrow{i_X} \mathbf{X} \xrightarrow{p_X} Z$ to $Z \xrightarrow{i_{X'}} \mathbf{X'} \xrightarrow{p_{X'}} Z$ is a map $h$ such that the following diagram commutes

$$\begin{tikzcd}
& Z \arrow[dl, "i_X"'] \arrow[dr, "i_{X'}"] & \\
\mathbf{X} \arrow[rr, "\mathbf{h}"] \arrow[dr, "p_X"'] & & \mathbf{X'}. \arrow[dl, "p_{X'}"] \\
& Z &
\end{tikzcd}$$

Clearly $Z$ is both the initial and terminal object in $\mathbf{C}_Z$ (i.e. the zero object). We will call $Z$ the base-object in $\mathbf{C}_Z$. In an attempt to reduce notational clutter, we generally suppress the maps to and from $Z$ and denote an object in $\mathbf{C}_Z$ just by $X \in \mathbf{C}_Z$. We will now show that the properties **P.1 - P.5** imposed on the ambient category ensure that the slice-coslice category has the following analogous properties **P.1' - P.5'**.

**P.1'** **$\mathbf{C}_Z$ has finite limits and finite colimits;**

**P.2'** **Products preserve pushouts in $\mathbf{C}_Z$;**

**P.3'** **$\mathbf{C}_Z$ has an interval object $I_z := I \times Z$ such that a factorisation of the fold morphism $S^0_z := S^0 \times Z \longrightarrow Z$ via the morphism $S^0_z \longrightarrow I_z$ exists:**

$$\begin{tikzcd} S^0_Z \arrow[rr, "id_Z \vee_Z id_Z"] \arrow[dr, dashed, "c_Z"'] & & Z; \\ & I_Z \arrow[ur, "\pi_{I_Z}"'] & \end{tikzcd}$$

**P.4' The morphism $c_Z : S^0_Z \longrightarrow I_Z$ is a cofibration in $\mathbf{C}_Z$;**

**P.5' The transposition morphism $\tau : S^0_Z \longrightarrow S^0_Z$ in $\mathbf{C}_Z$ extends to the commutative square**

$$\begin{tikzcd} S^0_Z \arrow[r, "c_Z"] \arrow[d, "\tau"'] & I_Z \arrow[d, dashed, "\tau"] \\ S^0_Z \arrow[r, "c_Z"'] & I_Z, \end{tikzcd}$$

**so that $\tau^2 = id$ in $\mathbf{C}_Z$.**

$\mathbf{C}_Z$ has a monoidal structure given by the product. However, there is a second, more useful, monoidal structure on $\mathbf{C}_Z$ which will be given by the **smash product**. We will define the smash product shortly, and prove that, in particular, it distributes over the coproduct. We now look at the structure and characteristics of $\mathbf{C}_Z$, and show that **P.1' - P.5'** hold.

**3.2.1 Lemma.** *Let $F : \mathbf{C}_Z \to \mathbf{C}$ be the forgetful functor. Consider a diagram*

$$G : \mathbf{I} \to \mathbf{C}_Z$$

*where $\mathbf{I}$ is a finite category. Let $\mathbf{I}_*$ be $\mathbf{I}$ with a final object $*$ added, and $\mathbf{I}^*$ be $\mathbf{I}$ with an initial object $*$ added.*

*(i) The limit of the diagram $G$ in $\mathbf{C}_Z$ is given by*

$$\lim G := \lim(F \circ G)_*$$

*where $(F \circ G)_* : \mathbf{I}_* \to \mathbf{C}$ is defined by $(F \circ G)_*|_{\mathbf{I}} = F \circ G$, and for $i \in \mathbf{I}$, the morphism $(F \circ G)_*(i \to *) = F \circ G(i) \to Z$ is the morphism to the base-object $Z$.*

*(ii) Dually, the colimit of the diagram $G$ in $\mathbf{C}_Z$ is given by*

$$\operatorname{colim} G := \operatorname{colim}(F \circ G)^*$$

*where $(F \circ G)^* : \mathbf{I}^* \to \mathbf{C}$ is defined by $(F \circ G)^*|_{\mathbf{I}} = F \circ G$, and for $i \in \mathbf{I}$, the morphism $(F \circ G)^*(* \to i) = Z \to F \circ G(i)$ is the morphism from the base-object $Z$.*

*Proof.* (i) The morphism
$$Z \to \lim G$$
is constructed by applying the universal property in $\mathbf{C}$ to the cone consisting of the morphisms
$$Z \to F \circ G(i)$$
for $i \in \mathbf{I}$. The morphism
$$\lim G \to Z$$
is the morphism to $(F \circ G)_*(*) = Z$ in the universal cone. One can then check that this defines the limit of diagram $G$ in $\mathbf{C}_Z$ with the required universal property.

(ii) The morphism
$$Z \to \operatorname{colim} G$$
is the morphism from $(F \circ G)^*(*)$ in the universal cocone. The morphism
$$\operatorname{colim} G \to Z$$
is obtained by applying the universal property of $\operatorname{colim}(F \circ G)^*$ in $\mathbf{C}$ to the cocone consisting of the morphisms
$$(F \circ G)(i) \to Z \text{ for } i \in \mathbf{I}.$$
□

**3.2.2 Example.**

1. The **product** in $\mathbf{C}_Z$ is the fibre product $X \times_Z Y$ in $\mathbf{C}$. The **coproduct** in $\mathbf{C}_Z$, known as the **wedge sum**, is the pushout $X \vee_Z Y$ in $\mathbf{C}$.

2. The **fibre products** and **pushouts** in $\mathbf{C}_Z$ are the same as those in $\mathbf{C}$ (when considered as objects of $\mathbf{C}$). So the standard properties of fibre products (Remark 3.1.4) and pushouts (Remark 3.1.2) also hold in $\mathbf{C}_Z$.

The pushout of $Z \xleftarrow{p_X} X \xrightarrow{f} Y$ in $\mathbf{C}_Z$ will be denoted by $Y/X$ , i.e.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d] \\ Z \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y/X. \end{tikzcd}$$

The (based) 0-sphere in $\mathbf{C}_Z$ is defined by taking the product of $S^0$ with $Z$ in $\mathbf{C}$. So $S^0_Z = Z \times S^0 = Z \vee_\emptyset Z$, i.e.

$$\begin{tikzcd} \emptyset \arrow[r, "\iota_Z"] \arrow[d, "\iota_Z"'] & Z \arrow[d, "i_0"] \\ Z \arrow[r, "i_1"'] \arrow[ur, phantom, "\ulcorner"] & S^0_Z. \end{tikzcd}$$

Note that in $\mathbf{C}_Z$ there exist two (based) 0-spheres, $(S^0_Z, i_0)$ and $(S^0_Z, i_1)$. Fix $S^0_Z$ to be $(S^0_Z, i_0)$.

**3.2.3 Lemma.** $\mathbf{C}_Z$ *satisfies properties **P.1'** - **P.5'**.*

*Proof.*

P.1' $\mathbf{C}_Z$ has finite limits and finite colimits by Lemma 3.2.1 and the fact that $\mathbf{C}$ has finite limits and colimits from **P.1**.

P.2' Products preserve pushouts in $\mathbf{C}_Z$ by Lemma 3.2.1 and the fact that fibre products preserve finite colimits in $\mathbf{C}$ from **P.2**.

P.3' The interval object in $\mathbf{C}_Z$ is obtained by taking the product of $I \in \mathbf{C}$ with the base-object $Z$. It will be denoted by $I_Z := I \times Z$. There exists a factorisation of the fold morphism $S^0_Z := S^0 \times Z \longrightarrow Z$ via the morphism $S^0_Z \longrightarrow I_Z$

$$\begin{tikzcd} S^0_Z \arrow[rr, "id_Z \vee_Z id_Z"] \arrow[dr, dashed, "c_Z"'] & & Z \\ & I_Z \arrow[ur, "\pi_{I_Z}"'] & \end{tikzcd}$$

obtained from the factorisation of the fold morphism $S^0 \to *$ via $c_* : S^0 \to I$ in **P.3** by taking the product with $Z$;

P.4' The fact that morphism $c_Z : S^0_Z \longrightarrow I_Z$ is a cofibration in $\mathbf{C}_Z$ will be proved in Lemma 3.2.23.

P.5' The transposition morphism $\tau : S^0_Z \longrightarrow S^0_Z$ in $\mathbf{C}_Z$ is just the product of the transposition morphism in $\mathbf{C}$ (as in **P.5**) with $Z$, and so extends to the commutative square

$$\begin{tikzcd} S^0_Z \arrow[r, "c_Z"] \arrow[d, "\tau"'] & I_Z \arrow[d, dashed, "\tau"] \\ S^0_Z \arrow[r, "c_Z"'] & I_Z, \end{tikzcd}$$

so that $\tau^2 = id$ in $\mathbf{C}_Z$.

□

**3.2.4 Remark.** For any $X \in \mathbf{C}_Z$, we have $X \times_Z Z \cong X$ and $X \vee_Z Z \cong X$. Then, by **P.2'**, for any $W, X, Y \in \mathbf{C}_Z$ we have

$$W \times_Z (X \vee_Z Y) \cong (W \times_Z X) \underset{W \times_Z Z}{\vee} (W \times_Z Y) \cong (W \times_Z X) \vee_W (W \times_Z Y)$$

**3.2.5 Definition.** The **smash product** of $X$ and $Y$ in $\mathbf{C}_Z$, denoted $X \wedge_Z Y$, is defined via the pushout

$$\begin{tikzcd}
X \vee_Z Y \arrow[r, "w"] \arrow[d] \arrow[dr, phantom, "\ulcorner" near end] & X \times_Z Y \arrow[d] \\
Z \arrow[r] & X \wedge_Z Y.
\end{tikzcd}$$

**3.2.6 Lemma** (Properties of the smash product, $\wedge_Z$, in $\mathbf{C}_Z$)**.**

*(i) Smash product is commutative (up to natural isomorphism).*

*(ii) The annihilating element for smash product is $Z$.*

*(iii) The unit element for smash product is $S^0_Z$.*

*(iv) Smash product is associative (up to natural isomorphism).*

*(v) Smash product preserves pushouts.*

*Proof.*

(i) This follows immediately from the commutativity of product and wedge. Consider

$$\begin{tikzcd}
X \vee_Z Y \arrow[rr] \arrow[dr, "\cong"] \arrow[dd] & & X \times_Z Y \arrow[dr, "\cong"] \arrow[dd] & \\
& Y \vee_Z X \arrow[rr] \arrow[dd] & & Y \times_Z X \arrow[dd] \\
Z \arrow[rr] \arrow[dr, equal] & & X \wedge_Z Y \arrow[dr, dashed, "\cong"] & \\
& Z \arrow[rr] & & Y \wedge_Z X
\end{tikzcd}$$

(where the back and front faces are the pushout squares defining $X \wedge_Z Y$ and $Y \wedge_Z X$ respectively). Clearly $X \wedge_Z Y \cong Y \wedge_Z X$.

(ii) Given any $X \in \mathbf{C}_Z$, we prove that $Z \wedge_Z X \cong Z$. Consider the pushout square defining the smash product of $X$ and $Z$:

$$\begin{array}{ccc} Z \vee_Z X & \xrightarrow{w} & Z \times_Z X \\ {\scriptstyle p}\downarrow & \ulcorner & \downarrow \\ Z & \longrightarrow & Z \wedge_Z X. \end{array}$$

Then, since $Z \times_Z X \cong X$ and $Z \vee_Z X \cong X$, we can rewrite the above as

$$\begin{array}{ccc} X & \longrightarrow & X \\ \downarrow & \ulcorner & \downarrow \\ Z & \longrightarrow & Z. \end{array}$$

Thus $Z \wedge_Z X \cong Z$. By (i) we have $X \wedge_Z Z \cong Z \wedge_Z X \cong Z$.

(iii) Given any $X \in \mathbf{C}_Z$, we prove that $S^0_Z \wedge_Z X \cong X$. The smash product of $X$ and $S^0$ is defined via the pushout

$$\begin{array}{ccc} S^0_Z \vee_Z X & \xrightarrow{w} & S^0_Z \times_Z X \\ {\scriptstyle p}\downarrow & \ulcorner & \downarrow \\ Z & \longrightarrow & S^0_Z \wedge_Z X \end{array}$$

where $S^0_Z \cong Z \vee_\emptyset Z$. Since $(Z \vee_\emptyset Z) \vee_Z X \cong X \vee_\emptyset Z$, and $(Z \vee_\emptyset Z) \times_Z X \cong X \vee_\emptyset X$ (by **P.2'**), the above pushout can be rewritten as

$$\begin{array}{ccc} X \vee_\emptyset Z & \xrightarrow{w} & X \vee_\emptyset X \\ {\scriptstyle p}\downarrow & \ulcorner & \downarrow \\ Z & \longrightarrow & X. \end{array}$$

Hence $S^0_Z \wedge_Z X \cong X$. Then $X \wedge_Z S^0_Z \cong S^0_Z \wedge_Z X \cong X$ by (i).

(iv) In order to prove that smash product is associative, we show that the following diagram is a pushout

$$\begin{array}{ccc} (A \times_Z B \times_Z Z) \vee_\emptyset (A \times_Z Z \times_Z C) \vee_\emptyset (Z \times_Z B \times_Z C) & \longrightarrow & Z \\ \downarrow & 1 & \downarrow \\ A \times_Z B \times_Z C & \longrightarrow & (A \wedge_Z B) \wedge_Z C, \end{array}$$

then the associativity of the smash product will follow since the rest of the diagram is clearly insensitive to order. Note that

$$(A \times_Z B) \vee_\emptyset (A \times_Z C) \vee_\emptyset (B \times_Z C) \cong (A \times_Z B \times_Z Z) \vee_\emptyset (A \times_Z Z \times_Z C) \vee_\emptyset (Z \times_Z B \times_Z C).$$

Consider the diagram

$$\begin{tikzcd}
(A \times_Z C) \vee_C (B \times_Z C) \arrow[dddd] \arrow[rrrr] & & & & C \arrow[dddd] \\
& A \vee_Z B \arrow[ul] \arrow[r] \arrow[d] \arrow[dr, phantom, "3"] & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "4"] & C \arrow[ur, equal] \arrow[d] & \\
\arrow[r, phantom, "2"] & A \times_Z B \arrow[r] \arrow[dl] & A \wedge_Z B \arrow[r] & (A \wedge_Z B) \vee_Z C \arrow[dr, "\cong"] & \\
& & & & \\
(A \times_Z B) \vee_\emptyset (A \times_Z C) \vee_\emptyset (B \times_Z C) \arrow[rrrr] & & & & \ulcorner.
\end{tikzcd}$$

Square 3 is the pushout defining the smash product of $A$ and $B$, and square 4 is the pushout defining the wedge of $(A \wedge_Z B)$ and $C$. Hence the inner rectangle is a pushout by the pasting law. Square 2 is also a pushout (by careful examination of the definitions of disjoint union, product and wedge). Thus the outer rectangle is a pushout by the pasting law.

Note that $(A \vee_Z B) \times_Z C \cong (A \times_Z C) \vee_C (B \times_Z C)$ and consider the following diagram:

$$\begin{tikzcd}
(A \vee_Z B) \times_Z C \arrow[r] \arrow[d] & C \arrow[d] \\
(A \times_Z B) \vee_\emptyset (A \times_Z C) \vee_\emptyset (B \times_Z C) \arrow[r] \arrow[d] \arrow[dr, phantom, "5"] & (A \wedge_Z B) \vee_Z C \arrow[d] \\
A \times_Z B \times_Z C \arrow[r] & (A \wedge_Z B) \times_Z C.
\end{tikzcd}$$

The top square is the pushout obtained above. The outer rectangle is the pushout obtained from the pushout defining $A \wedge_Z B$ by taking the product with $C$ (by **P.2'**). Thus, by the pasting law, square 5 is a pushout. Then, considering

$$\begin{tikzcd}
(A \times_Z B \times_Z Z) \vee_\emptyset (A \times_Z Z \times_Z C) \vee_\emptyset (Z \times_Z B \times_Z C) \arrow[r] \arrow[d] \arrow[dr, phantom, "5"] & (A \wedge_Z B) \vee_Z C \arrow[r] \arrow[d] \arrow[dr, phantom, "6"] & Z \arrow[d] \\
A \times_Z B \times_Z C \arrow[r] & (A \wedge_Z B) \times_Z C \arrow[r] & (A \wedge_Z B) \wedge_Z C,
\end{tikzcd}$$

we obtain square 1 as the outer rectangle. Since 5 is a pushout square and 6 is the pushout square defining $(A \wedge_Z B) \wedge_Z C$, by the pasting law, square 1 is a pushout, as required. Therefore

$$(A \wedge_Z B) \wedge_Z C \cong A \wedge_Z (B \wedge_Z C).$$

(v) Suppose that $X \in \mathbf{C}_Z$. We take $A, B, C, D$ to be objects in $\mathbf{C}_Z$ and assume that the diagram below is a pushout square in $\mathbf{C}_Z$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & D.
\end{tikzcd}$$

We know that by construction this comes from the corresponding colimit diagram in $\mathbf{C}$:

$$\begin{tikzcd}
Z \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& A \arrow[r] \arrow[d] & B \arrow[d] \\
& C \arrow[r] \arrow[ur, phantom, "c" very near end] & D.
\end{tikzcd}$$

By **P.2**, the following is also a colimit diagram in $\mathbf{C}$:

$$\begin{tikzcd}
Z \times X \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A \times X \arrow[r] \arrow[d] & B \times X \arrow[d] \\
& C \times X \arrow[r] \arrow[ur, phantom, "c" very near end] & D \times X.
\end{tikzcd} \tag{3.2.1}$$

Let $E \in \mathbf{C}_Z$ be the object defined by the following pushout in $\mathbf{C}_Z$:

$$\begin{tikzcd}
A \wedge_z X \arrow[r] \arrow[d] & B \wedge_z X \arrow[d] \\
C \wedge_z X \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & E.
\end{tikzcd}$$

Then we need to prove that $E \cong D \wedge_z X$. As an unbased object, $D \wedge_z X$ is the colimit of diagram 3.2.1 with the morphism $Z \times X \to Z$ added:

$$D \wedge_z X \cong \operatorname{colim}\left(\begin{tikzcd}
Z \times X \arrow[r, bend left] \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & Z & \\
& A \times X \arrow[r] \arrow[d] & B \times X \\
& C \times X &
\end{tikzcd}\right).$$

The proof that

$$D \wedge_z X \cong \operatorname{colim}\left(\begin{tikzcd}
A \wedge_z X \arrow[r] \arrow[d] & B \wedge_z X \\
C \wedge_z X &
\end{tikzcd}\right)$$

is done in two stages, using the notion of a *cofiltered* category. A category $\mathbf{I}$ is said to be cofiltered if and only if

(a) $\mathbf{I} \neq \emptyset$;

(b) Given $i, i' \in \mathbf{I}$, there exists $i'' \in \mathbf{I}$ and dotted morphisms as below:

$$\begin{tikzcd} i \arrow[dr, dashed] & \\ & i''; \\ i' \arrow[ur, dashed] & \end{tikzcd}$$

(c) Given parallel morphisms from $i$ to $i'$ there exists $i'' \in I$ and a dotted morphism, as indicated, which coequalises them: $i \rightrightarrows i' \dashrightarrow i''$.

The first stage is to show that colimits can be added in step-by-step, so that:

$$\mathrm{colim}\left(\begin{tikzcd} Z \times X \arrow[r, bend left] \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & Z & \\ & A \times X \arrow[r] \arrow[d] & B \times X \\ & C \times X & \end{tikzcd}\right) \cong \mathrm{colim}\left(\begin{tikzcd} Z \times X \arrow[rrr, bend left] \arrow[dr] \arrow[drrr, bend left] \arrow[dddr, bend right] & & & Z & \\ & A \times X \arrow[rr] \arrow[dr] \arrow[dd] & & B \times X \arrow[dr] & \\ & & A \wedge_z X \arrow[rr] \arrow[dd] & & B \wedge_z X \\ & C \times X \arrow[dr] & & & \\ & & C \wedge_z X & & \end{tikzcd}\right).$$

These two diagrams can be viewed as cofiltered categories, $\mathbf{J} \to \mathbf{I}$. The proof that they induce the same colimits follows from the fact that, given $\mathbf{J} \to \mathbf{I}$ and $F : \mathbf{I} \longrightarrow \mathbf{C}$, if $F(\mathbf{I})'$ is diagram $F(\mathbf{I})$ with one object $\mathrm{colim}_{\mathbf{J}} F$ added, together with canonical morphisms $F(j) \longrightarrow \mathrm{colim}\, F(\mathbf{J})$ for $j \in \mathbf{J}$, then $\mathrm{colim}\, F(\mathbf{I})' \cong \mathrm{colim}\, F(\mathbf{I})$.

The second stage is to show that under certain conditions, colimits can be deleted:

$$\mathrm{colim}\left(\begin{tikzcd} Z \times X \arrow[rrr, bend left] \arrow[dr] \arrow[drrr, bend left] \arrow[dddr, bend right] & & & Z & \\ & A \times X \arrow[rr] \arrow[dr] \arrow[dd] & & B \times X \arrow[dr] & \\ & & A \wedge_z X \arrow[rr] \arrow[dd] & & B \wedge_z X \\ & C \times X \arrow[dr] & & & \\ & & C \wedge_z X & & \end{tikzcd}\right) \cong \mathrm{colim}\left(\begin{tikzcd} A \wedge_z X \arrow[r] \arrow[d] & B \wedge_z X \\ C \wedge_z X & \end{tikzcd}\right).$$

Viewing the two diagrams as cofiltered categories, this follows from the fact that, given $\mathbf{J} \to \mathbf{I}$ such that

(a) for each $j \in J$ there is some $i \in \mathbf{I}$ and a morphism $i \dashrightarrow j$ in $\mathbf{I}$;

(b) for a solid diagram as below there exists a morphism $i \to j''$ in $\mathbf{I}$ such that the diagram below commutes:

$$\begin{tikzcd} & & j \\ i \arrow[r, dashed] \arrow[urr, bend left] \arrow[drr, bend right] & j'' \arrow[ur] \arrow[dr] & & \text{in } \mathbf{I}, \\ & & j' \end{tikzcd}$$

then $\operatorname{colim}_{\mathbf{I}} \cong \operatorname{colim}_{\mathbf{J}}$ .

□

**3.2.7 Corollary.** *The smash product distributes over the wedge sum, i.e. for any $A, B, C \in \mathbf{C}_Z$, we have*

$$(A \vee_Z B) \wedge_Z C \cong (A \wedge_Z C) \vee_Z (B \wedge_Z C).$$

*Proof.* This follows from Lemma 3.2.6. By (v) we have

$$(A \vee_Z B) \wedge_Z C \cong (A \wedge_Z C) \vee_{Z \wedge_Z C} (B \wedge_Z C).$$

Then, since $Z \wedge_Z C \cong Z$ by (ii),

$$(A \vee_Z B) \wedge_Z C \cong (A \wedge_Z C) \vee_Z (B \wedge_Z C).$$

□

**3.2.8 Corollary.** *$\mathbf{C}_Z$ is a monoidal category.*

*Proof.* This is a direct consequence of Lemma 3.2.6. The monoidal structure on $\mathbf{C}_Z$ is as follows:

(i) the functor $\_ \wedge_Z \_ : \mathbf{C}_Z \times \mathbf{C}_Z \to \mathbf{C}_Z$ is the tensor product;

(ii) the smash product is associative (by Lemma 3.2.6, (iv)), i.e, for any $A, B, C \in \mathbf{C}_Z$ there exists a natural isomorphism

$$(A \wedge_Z B) \wedge_Z C \xrightarrow[\cong]{a} A \wedge_Z (B \wedge_Z C);$$

(iii) the object $S^0_Z \in \mathbf{C}_Z$ is the unit object and for any $A \in \mathbf{C}_Z$ there exist natural isomorphisms $S^0_Z \wedge_Z A \xrightarrow[\cong]{u_L} A$ and $A \wedge_Z S^0_Z \xrightarrow[\cong]{u_R} A$ (by Lemma 3.2.6, (iii));

such that

(a) for any $B \in \mathbf{C}_Z$

$$\begin{tikzcd}
(A \wedge_Z S^0_Z) \wedge_Z B \arrow[rr, "a_{A,S^0_Z,B}"] \arrow[dr, "u_R \wedge_Z id_B"'] & & A \wedge_Z (S^0_Z \wedge_Z B) \arrow[dl, "id_A \wedge_Z u_L"] \\
& A \wedge_Z B &
\end{tikzcd}$$

commutes,

(b) for any $D \in \mathbf{C}_Z$

$$\begin{tikzcd}
((A \wedge_Z B) \wedge_Z C) \wedge_Z D \arrow[r, "a \wedge_Z id_D"] \arrow[d, "a"'] & (A \wedge_Z (B \wedge_Z C)) \wedge_Z D \arrow[r, "a"] & A \wedge_Z ((B \wedge_Z C) \wedge_Z D) \arrow[d, "id_A \wedge_Z a"] \\
(A \wedge_Z B) \wedge_Z (C \wedge_Z D) \arrow[rr, "a"'] & & A \wedge_Z (B \wedge_Z (C \wedge_Z D))
\end{tikzcd}$$

commutes.

□

Consider the forgetful functor $F : \mathbf{C}_Z \to \mathbf{C}/Z$ where $\mathbf{C}/Z$ is the slice category of $\mathbf{C}$ over $Z$. The left adjoint to $F$ is the functor

$$(\_)^+ := (\_) \vee_\emptyset Z : \mathbf{C}/Z \to \mathbf{C}_Z.$$

Thus $(\_)^+ : \mathbf{C}/Z \to \mathbf{C}_Z$ preserves colimits since it is left adjoint (so right exact), and $F : \mathbf{C}_Z \to \mathbf{C}/Z$ preserves limits since it is right adjoint (so left exact). We note that

$$(X \vee_\emptyset Y)^+ \cong X^+ \vee_Z Y^+,$$

and

$$(X \times Y)^+ \cong X^+ \wedge_Z Y^+.$$

There is an appropriate (based) interval object $I_Z = I \times Z$ in the slice-coslice category as in **P.3'**, where the morphism $c_Z : S^0_Z \to I_Z$ in $\mathbf{C}_Z$ is induced from $c_* : S^0 \to I$ (since taking the product with $Z$ in $\mathbf{C}$ preserves pushouts by **P.2**). We denote the composites

$$Z \xrightarrow{i_0} S^0_Z \xrightarrow{c_Z} I_Z$$

and

$$Z \xrightarrow{i_1} S^0_Z \xrightarrow{c_Z} I_Z$$

by $i_0 : Z \to I_Z$ and $i_1 : Z \to I_Z$ respectively. We use the notation $0 := i_0(Z) \in I_Z$ and $1 := i_1(Z) \in I_Z$. Thus there exist two (based) interval objects $(I_Z, i_0)$ and

$(I_Z, i_1)$ in $\mathbf{C}_Z$. We fix the standard (based) interval in $\mathbf{C}_Z$ to be $(I_Z, i_0)$. We also write $c_0 : (S^0_Z, i_0) \to (I_Z, i_0)$ and $c_1 : (S^0_Z, i_0) \to (I_Z, i_1)$.

An important object in $\mathbf{C}_Z$ is $I^+_Z$, the interval $I_Z$ with a disjoint base-object adjoined:

$$I^+_Z = (I_Z)^+ := I_Z \vee_\emptyset Z = (I \times Z) \vee_\emptyset Z,$$

i.e. $I^+_Z$ is defined via the pushout

$$\begin{tikzcd}
\emptyset \arrow[r, "\iota_Z"] \arrow[d, "\iota_{I_Z}"'] \arrow[dr, phantom, "\ulcorner", very near end] & Z \arrow[d, "i_{I^+_Z}"] \\
I_Z \arrow[r, "\upsilon"'] & I^+_Z
\end{tikzcd}$$

We denote the inclusion of the basepoint into $I^+_Z$ by $i_{I^+_Z} : Z \to I^+_Z$. Consider $S^0_Z \to I^+_Z$ and denote by $\upsilon_0$ and $\upsilon_1$, the maps which take the base-object of $S^0_Z$ to the disjoint base-object in $I^+_Z$ and the remaining copy of $Z$ in $S^0_Z$ to the 0 and 1 ends of the interval in $I^+_Z$, respectively.

**3.2.9 Definition.** Consider any object $X$ in $\mathbf{C}_Z$.

1. The **(reduced) cylinder** on $X$ is defined to be $\mathrm{Cyl}_Z(X) := X \wedge_Z I^+_Z \in \mathbf{C}_Z$.

2. The **(reduced) cone** on $X$ is defined to be $\mathrm{C}_Z(X) := X \wedge_Z I_Z \in \mathbf{C}_Z$.

**3.2.10 Definition.** Consider any morphism $f : X \to Y$ in $\mathbf{C}_Z$.

1. The **mapping cylinder** of $f$, denoted by $\mathrm{Cyl}_Z(f)$ in $\mathbf{C}_Z$, is defined via the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id_X \wedge_Z \upsilon_1"'] \arrow[dr, phantom, "\ulcorner", very near end] & Y \arrow[d] \\
\mathrm{Cyl}_Z(X) \arrow[r] & \mathrm{Cyl}_Z(f).
\end{tikzcd}$$

2. The **mapping cone** of $f$, denoted by $\mathrm{C}_Z(f)$ in $\mathbf{C}_Z$, is defined via the pushout

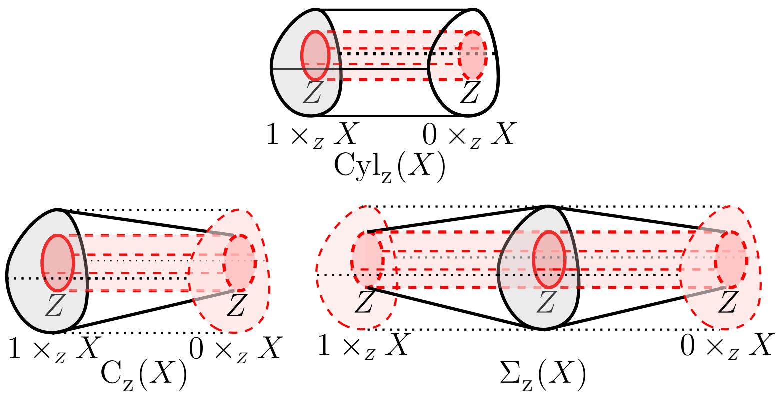
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id_X \wedge_Z c_1"'] \arrow[dr, phantom, "\ulcorner", very near end] & Y \arrow[d, "f_1"] \\
\mathrm{C}_Z(X) \arrow[r] & \mathrm{C}_Z(f).
\end{tikzcd}$$

**3.2.11 Definition.** The **(reduced) suspension** of $X$, denoted by $\Sigma_Z(X)$, is defined to be the mapping cone of $p_X : X \to Z$, i.e. the pushout

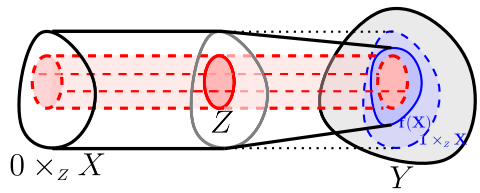
$$\begin{tikzcd} X \arrow[r, "p_X"] \arrow[d, "id_X \wedge_Z c_1"'] \arrow[dr, phantom, "\ulcorner", very near end] & Z \arrow[d] \\ \mathrm{C_z}(X) \arrow[r, "k"'] & \Sigma_{\mathrm{z}}(X). \end{tikzcd}$$

So $\Sigma_{\mathrm{z}}(X) = \mathrm{C_z}(p_x)$.

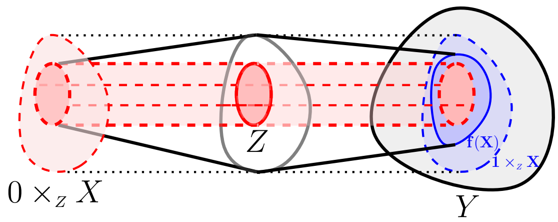
**3.2.12 Remark.** The cylinder, cone and suspension of an object $X$ in $\mathbf{C}_Z$ can be visualised as below. The components highlighted in red in the diagrams indicate the points which get identified to a single copy of $Z$.




Similarly, the mapping cylinder and the mapping cone of $f : X \to Y$ can be pictured as below:




The mapping cylinder of $f : X \to Y$.




The mapping cone of $f : X \to Y$.

**3.2.13 Example.** The 1-sphere, another important object in $\mathbf{C}_Z$, is defined as $S^1_Z := \Sigma_{\mathrm{z}}(S^0_Z)$, i.e.

$$\begin{tikzcd} S^0_Z \arrow[r, "p_{S^0_Z}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & Z \arrow[d] \\ \mathrm{C_z}(S^0_Z) \arrow[r] & \Sigma_{\mathrm{z}}(S^0_Z), \end{tikzcd}$$

where $\mathrm{C_z}(S^0_{\scriptscriptstyle Z}) = S^0_{\scriptscriptstyle Z} \wedge_Z I_{\scriptscriptstyle Z} = I_{\scriptscriptstyle Z}$. So, in fact the 1-sphere is also described by the following pushout

$$\begin{tikzcd} S^0_{\scriptscriptstyle Z} \arrow[r, "p_{S^0_Z}"] \arrow[d, "c_Z"'] \arrow[dr, phantom, "\ulcorner" very near end] & Z \arrow[d] \\ I_{\scriptscriptstyle Z} \arrow[r] & S^1_{\scriptscriptstyle Z} \end{tikzcd}$$

in which the two 'ends' of the interval $I_{\scriptscriptstyle Z}$ are identified.

The $n$**-sphere** in $\mathbf{C}_Z$ is defined as

$$S^n_{\scriptscriptstyle Z} := \Sigma^{\mathrm{n}}_{\mathrm{z}}(S^0_{\scriptscriptstyle Z}).$$

We note that higher dimensional spheres in the ambient category $\mathbf{C}$ can be constructed in $\mathbf{C}_*$ (where $*$ is the terminal object in $\mathbf{C}$) via the smash product before then applying the base-object-forgetting functor.

**3.2.14 Lemma.** *Given any $X \in \mathbf{C}_Z$, we have*

$$\Sigma^{\mathrm{n}}_{\mathrm{z}}(X) \cong X \wedge_Z S^n_{\scriptscriptstyle Z}.$$

*Proof.* Considering the pushout square defining $S^1_{\scriptscriptstyle Z}$, and taking the smash product with $X \in \mathbf{C}_Z$, we obtain

$$\begin{tikzcd} S^0_{\scriptscriptstyle Z} \wedge_Z X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & Z \wedge_Z X \arrow[d] \\ I_{\scriptscriptstyle Z} \wedge_Z X \arrow[r] & S^1_{\scriptscriptstyle Z} \wedge_Z X. \end{tikzcd}$$

Since $Z \wedge_Z X \cong Z$, and $S^0_{\scriptscriptstyle Z} \wedge_Z X \cong X$, this can be rewritten as

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & Z \arrow[d] \\ I_{\scriptscriptstyle Z} \wedge_Z X \arrow[r] & \Sigma_{\mathrm{z}}(X), \end{tikzcd}$$

which is precisely the pushout square defining the reduced suspension of $X$. Hence $\Sigma_{\mathrm{z}}(X) \cong X \wedge_Z S^1_{\scriptscriptstyle Z}$. Inductively we have $\Sigma^{\mathrm{n}}_{\mathrm{z}}(X) \cong X \wedge_Z S^n_{\scriptscriptstyle Z}$. $\square$

We make the following remark regarding the existence of the morphism $\bar{c_Z}$ : $X \vee_Z X \longrightarrow I^+_{\scriptscriptstyle Z} \wedge_Z X$ which is required in order to define *based homotopy* in $\mathbf{C}_Z$

**3.2.15 Remark.** Consider the morphism $c_Z : S^0_{\scriptscriptstyle Z} \to I_{\scriptscriptstyle Z}$ as in **P.3**. Applying the functor $(\_)^+$ and taking the smash product with $X \in \mathbf{C}_Z$, by Lemma 3.2.6, (v), we obtain

$$\bar{c_Z} : (S^0_{\scriptscriptstyle Z})^+ \wedge_Z X \longrightarrow I^+_{\scriptscriptstyle Z} \wedge_Z X = \mathrm{Cyl_z}(X).$$

We note that $(S^0_Z)^+ \cong S^0_Z \vee_\emptyset Z \cong S^0_Z \vee_Z S^0_Z$, and thus

$$(S^0_Z)^+ \wedge_Z X \cong (S^0_Z \vee_Z S^0_Z) \wedge_Z X \cong (S^0_Z \wedge_Z X) \vee_Z (S^0_Z \wedge_Z X) \cong X \vee_Z X.$$

So there exists a morphism $\bar{c_Z} : X \vee_Z X \longrightarrow I^+_Z \wedge_Z X$.

We can now define the notion of based homotopy in $\mathbf{C}_Z$.

**3.2.16 Definition.** Given two morphisms $f, g : X \to Y$ in $\mathbf{C}_Z$, a **(based) homotopy** from $f$ to $g$ is a morphism $h : X \wedge_Z I^+_Z \to Y$ such that the following diagram commutes

$$\begin{tikzcd} X \vee_Z X \arrow[rr, "f \vee_Z g"] \arrow[dr, "\bar{c_Z}"'] & & Y. \\ & X \wedge_Z I^+_Z \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

The morphisms $f$ and $g$ are **homotopic** in $\mathbf{C}_Z$, $f \simeq g$, if such a homotopy exists.

**3.2.17 Definition.** A morphism $f : X \to Y$ in $\mathbf{C}_Z$ is a **homotopy equivalence** if there exists a morphism $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. The objects $X$ and $Y$ in $\mathbf{C}_Z$ are then said to be **homotopy equivalent**, $X \simeq Y$.

**3.2.18 Lemma.** *For any $X \in \mathbf{C}_Z$, the smash product $\_ \wedge_Z X$ preserves*

*(i) (based) homotopy in $\mathbf{C}_Z$;*

*(ii) the mapping cone on a morphism $f : A \to B$ in $\mathbf{C}_Z$;*

*(iii) the suspension of an object $A \in \mathbf{C}_Z$.*

*Proof.* This follows from Lemma 3.2.6, in particular commutativity and associativity of the smash product, (i) and (iv).

(i) Given a homotopy $A \wedge_Z I^+_Z \to B$ in $\mathbf{C}_Z$, taking the smash product with $X \in \mathbf{C}_Z$, we have

$$(A \wedge_Z I^+_Z) \wedge_Z X \to B \wedge_Z X,$$

where $(A \wedge_Z I^+_Z) \wedge_Z X \cong (A \wedge_Z X) \wedge_Z I^+_Z$. Hence we obtain a homotopy

$$(A \wedge_Z X) \wedge_Z I^+_Z \to B \wedge_Z X.$$

(ii) Given a morphism $f : A \to B$ in $\mathbf{C}_Z$, the mapping cone on $f$ is defined to be the pushout along $A \xrightarrow{f} B$ and $A \xrightarrow{id \wedge_Z v_1} \mathrm{C}_Z(A)$ where $\mathrm{C}_Z(A) := A \wedge_Z I_Z$. Taking the smash product of $\mathrm{C}_Z(A)$ with $X \in \mathbf{C}_Z$, we obtain

$$\mathrm{C}_Z(A) \wedge_Z X \cong (A \wedge_Z I_Z) \wedge_Z X \cong (A \wedge_Z X) \wedge_Z I_Z \cong \mathrm{C}_Z(A \wedge_Z X).$$

Thus, since smash product preserves pushouts by (v), we have

$$\mathrm{C}_{\mathrm{z}}(f) \wedge_Z X \cong \mathrm{C}_{\mathrm{z}}(f \wedge_Z X)$$

where $f \wedge_Z X : A \wedge_Z X \to B \wedge_Z X$.

(iii) The suspension of an object $A \in \mathbf{C}_Z$ is defined to be $\Sigma_{\mathrm{z}}(A) := A \wedge_Z S^1_Z$. Taking the smash product with $X \in \mathbf{C}_Z$, we have

$$\begin{aligned}(\Sigma_{\mathrm{z}}(A)) \wedge_Z X &\cong (A \wedge_Z S^1_Z) \wedge_Z X \\ &\cong (A \wedge_Z X) \wedge_Z S^1_Z \\ &\cong \Sigma_{\mathrm{z}}(A \wedge_Z X).\end{aligned}$$

□

Now, having given meaning to the notion of (based) homotopy, we are able to consider cofibrations in $\mathbf{C}_Z$. The following result, stated as a Lemma, concerns the existence of a morphism $\mathrm{Cyl}_{\mathrm{z}}(A \to B) \to B \wedge_Z I^+_Z$ which will be used in the definition of a cofibration.

**3.2.19 Lemma.** *Given $f : A \to B$ in $\mathbf{C}_Z$ there exists a canonical morphism*

$$(A \wedge_Z I^+_Z) \underset{A \wedge_Z S^0_Z}{\vee} (B \wedge_Z S^0_Z) = \mathrm{Cyl}_{\mathrm{z}}(f) \xrightarrow{\quad i \quad} B \wedge_Z I^+_Z$$

*such that $i\,|_{A \wedge_Z I^+_Z} = f \wedge_Z id_{I^+_Z}$, and $i\,|_{B \wedge_Z S^0_Z} = id_B \wedge_Z c^+_Z$ where*

$$c^+_Z := (\_)^+ \circ c_Z : S^0_Z \longrightarrow I_Z \longrightarrow I^+_Z.$$

*Proof.* Note that the morphism $i_0 : Z \to I_Z$ in $\mathbf{C}_Z$ induces a morphism $i_0^+ : Z^+ \to I^+_Z$ where $Z^+ = S^0_Z$. Then, given $f : A \to B$ in $\mathbf{C}_Z$, the morphism $i : \mathrm{Cyl}_{\mathrm{z}}(f) \longrightarrow B \wedge_Z I^+_Z$ exists uniquely by the universal property of the pushout diagram

$$\begin{tikzcd}
A \wedge_Z S^0_Z \arrow[r, "f \wedge_Z id_{S^0_Z}"] \arrow[d, "id_A \wedge_Z i_0^+"'] & B \wedge_Z S^0_Z \arrow[d] \arrow[ddr, bend left, "id_B \wedge_Z c^+_Z"] & \\
A \wedge_Z I^+_Z \arrow[r] \arrow[drr, bend right, "f \wedge_Z id_{I^+_Z}"'] & (A \wedge_Z I^+_Z) \underset{A \wedge_Z S^0_Z}{\vee} (B \wedge_Z S^0_Z). \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists ! i"] & \\
 & & B \wedge_Z I^+_Z
\end{tikzcd}$$

where $c_Z^+ := (\_)^+ \circ c_Z : S_Z^0 \longrightarrow I_Z \longrightarrow I_Z^+$. $\square$

The morphism $i$ defined above is required for the following definition.

**3.2.20 Definition.** A morphism $f : A \to B$ in $\mathbf{C}_Z$ is a **(based) cofibration** if it has the (based) homotopy extension property, i.e. if there exists a morphism $G$ which completes the following diagram (not necessarily uniquely) to a commutative diagram:

$$\begin{tikzcd} \mathrm{Cyl}_Z(f) \arrow[r, "i"] \arrow[d] & B \wedge_Z I_Z^+. \arrow[dl, dashed, "\exists G"] \\ X & \end{tikzcd} \tag{3.2.2}$$

In other words, a morphism $f : A \to B$ is a cofibration in $\mathbf{C}_Z$ if, whenever there exists a morphism $j : B \wedge_Z S_Z^0 \to X$ and a homotopy $J : A \wedge_Z I_Z^+ \to X$ such that $J|_{A \wedge_Z S_Z^0} = j \circ (f \wedge_Z id_{S_Z^0})$, then the homotopy can be extended to $G : B \wedge_Z I_Z^+ \to X$ such that $G \circ (f \wedge_Z id_{I_Z^+}) = J$ and $G|_{B \wedge_Z S_Z^0} = j$.

**3.2.21 Lemma.** *A morphism $f : A \to B$ is a cofibration in $\mathbf{C}_Z$ if and only if $\mathrm{Cyl}_Z(f)$ is a retract of $B \wedge_Z I_Z^+$, i.e. there exist morphisms*

$$\mathrm{Cyl}_Z(f) \xrightarrow{\quad i \quad} B \wedge_Z I_Z^+ \xrightarrow{\quad r \quad} \mathrm{Cyl}_Z(f)$$

*such that $r \circ i = id$, then $r$ is a retraction of $B \wedge_Z I_Z^+$ onto $\mathrm{Cyl}_Z(f)$.*

*Proof.* Assume $f : A \to B$ is a cofibration in $\mathbf{C}_Z$. Then, considering

$$\begin{tikzcd} \mathrm{Cyl}_Z(f) \arrow[r, "i"] \arrow[d, "id"'] & B \wedge_Z I_Z^+ \arrow[dl, dashed, "\exists r"] \\ \mathrm{Cyl}_Z(f) & \end{tikzcd}$$

we obtain a retraction $r : B \wedge_Z I_Z^+ \longrightarrow \mathrm{Cyl}_Z(f)$. The opposite direction is obvious: given a retraction $B \wedge_Z I_Z^+ \longrightarrow \mathrm{Cyl}_Z(f)$, then clearly $f : A \to B$ is a cofibration. $\square$

**3.2.22 Lemma** (Properties of cofibrations in $\mathbf{C}_Z$)**.**

*(i) Cofibrations are closed under composition in $\mathbf{C}_Z$.*

*(ii) Cofibrations are closed under pushout in $\mathbf{C}_Z$.*

*(iii) Cofibrations are preserved by the wedge sum, $\vee_Z$, in $\mathbf{C}_Z$.*

*(iv) Cofibrations are preserved by the smash product, $\wedge_Z$, in $\mathbf{C}_Z$.*

*Proof.*

(i) Suppose $f : A \to B$ and $g : B \to C$ are cofibrations in $\mathbf{C}_Z$. We have

$$\mathrm{Cyl}_{\mathrm{z}}(g) = (B \wedge_Z I_Z^+) \vee_g C.$$

Consider the composite morphism $g \circ f : A \to C$. Then

$$\mathrm{Cyl}_{\mathrm{z}}(gf) = (A \wedge_Z I_Z^+) \vee_{gf} C.$$

By writing out the definitions of these mapping cylinders as pushouts, we obtain the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & A \wedge_Z I_Z^+ \arrow[d] & \\
B \arrow[r] \arrow[d, "g"'] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Cyl}_{\mathrm{z}}(f) \arrow[r, "i"] \arrow[d] & B \wedge_Z I_Z^+ \arrow[d] \\
C \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Cyl}_{\mathrm{z}}(gf) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Cyl}_{\mathrm{z}}(g).
\end{tikzcd}$$

The top square is the pushout defining $\mathrm{Cyl}_{\mathrm{z}}(f)$, the left hand rectangle is the pushout defining $\mathrm{Cyl}_{\mathrm{z}}(gf)$, thus by the pasting law for pushouts, the bottom left hand square is also a pushout. Then, again by the pasting law, since the bottom rectangle is the pushout defining $\mathrm{Cyl}_{\mathrm{z}}(g)$ and the bottom left hand square is a pushout, we have

$$\begin{tikzcd}
\mathrm{Cyl}_{\mathrm{z}}(f) \arrow[r, "i"] \arrow[d] & B \wedge_Z I_Z^+ \arrow[d] \\
\mathrm{Cyl}_{\mathrm{z}}(gf) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Cyl}_{\mathrm{z}}(g).
\end{tikzcd}$$

By the definition of a cofibration, given a morphism $\mathrm{Cyl}_{\mathrm{z}}(f) \to X$ there exists a morphism $F : B \wedge_Z I_Z^+ \to X$ making

$$\begin{tikzcd}
\mathrm{Cyl}_{\mathrm{z}}(f) \arrow[r] \arrow[d] & B \wedge_Z I_Z^+ \arrow[dl, dashed, "\exists F"] \\
X &
\end{tikzcd}$$

commute. Similarly, given $\mathrm{Cyl}_{\mathrm{z}}(g) \to X$, there exists $G : C \wedge_Z I_Z^+ \to X$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Cyl}_{\mathrm{z}}(g) \arrow[r] \arrow[d] & C \wedge_Z I_Z^+ \arrow[dl, dashed, "\exists G"] \\
X. &
\end{tikzcd}$$

Suppose there exists a morphism $\mathrm{Cyl}_{\mathrm{z}}(gf) \longrightarrow X$. Then to see that the composite $g \circ f$ is a cofibration, consider the following diagram

$$\begin{tikzcd}
\mathrm{Cyl}_{\mathrm{z}}(f) \arrow[r] \arrow[d] & B \wedge_Z I_Z^+ \arrow[d] \arrow[ddl, dashed, bend right, "F"'] & \\
\mathrm{Cyl}_{\mathrm{z}}(gf) \arrow[r] \arrow[d] & \mathrm{Cyl}_{\mathrm{z}}(g) \arrow[r] \arrow[dl, dotted, "\exists!"] \arrow[ul, phantom, "\ulcorner"] & C \wedge_Z I_Z^+, \arrow[dll, dashed, bend left, "G"] \\
X & &
\end{tikzcd}$$

where, by the universal property of pushouts, there exists a unique morphism $\mathrm{Cyl}_{\mathrm{z}}(g) \to X$.

(ii) Suppose $f : A \to B$ is a cofibration. Consider the pushout of $C \leftarrow A \xrightarrow{f} B$ for some $C$ in $\mathbf{C}_Z$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
C \arrow[r, "g"'] \arrow[ur, phantom, "\ulcorner"] & D.
\end{tikzcd}$$

In order to prove that $g : C \to D$ is also a cofibration, we need to show that the following square is a pushout:

$$\begin{tikzcd}
\mathrm{Cyl}_{\mathrm{z}}(f) \arrow[r] \arrow[d] & B \wedge_Z I_Z^+ \arrow[d] \\
\mathrm{Cyl}_{\mathrm{z}}(g) \arrow[r] & D \wedge_Z I_Z^+.
\end{tikzcd}$$

By Remark 3.1.2, (ii), the outer rectangle in the following diagram is a pushout:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
C \arrow[r, "g"'] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & D \arrow[d] \\
C \wedge_Z I_Z^+ \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Cyl}_{\mathrm{z}}(g).
\end{tikzcd}$$

Now consider

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A \wedge_Z I_Z^+ \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner", very near end] \arrow[dr, phantom, "1"] & \mathrm{Cyl}_Z(f) \arrow[d] \\
C \wedge_Z I_Z^+ \arrow[r] & \mathrm{Cyl}_Z(g)
\end{tikzcd}$$

where the top square is the pushout defining $\mathrm{Cyl}_Z(f)$ and the outer rectangle is a pushout by above. Thus, by the pasting law for pushouts the bottom square, 1, is also a pushout. Since smash product preserves pushouts (by Lemma 3.2.6, (v)), the following diagram is a pushout in $\mathbf{C}_Z$:

$$\begin{tikzcd}
A \wedge_Z I_Z^+ \arrow[r, "f \wedge_Z id"] \arrow[d] & B \wedge_Z I_Z^+ \arrow[d] \\
C \wedge_Z I_Z^+ \arrow[r, "g \wedge_Z id"'] \arrow[ur, phantom, "\ulcorner", very near end] & D \wedge_Z I_Z^+ .
\end{tikzcd}$$

Then, considering

$$\begin{tikzcd}
A \wedge_Z I_Z^+ \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \mathrm{Cyl}_Z(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "2"] & B \wedge_Z I_Z^+ \arrow[d] \\
C \wedge_Z I_Z^+ \arrow[r] & \mathrm{Cyl}_Z(g) \arrow[r] & D \wedge_Z I_Z^+ ,
\end{tikzcd}$$

by the pasting law, square 2 is a pushout, as required. Since $f$ is a cofibration, given a morphism $\mathrm{Cyl}_Z(f) \to X$, there exists a homotopy $H : B \wedge_Z I_Z^+ \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Cyl}_Z(f) \arrow[r] \arrow[d] & B \wedge_Z I_Z^+ \arrow[dl, dashed, "\exists H"] \\
X . &
\end{tikzcd}$$

Therefore, given a morphism $\mathrm{Cyl}_Z(g) \to X$, by the universal property of pushout 2, there exists a unique morphism $D \wedge_Z I_Z^+ \to X$ as below:

$$\begin{tikzcd}
\mathrm{Cyl}_Z(f) \arrow[r] \arrow[d] & B \wedge_Z I_Z^+ \arrow[d] \arrow[ddl, bend right, "H"'] \\
\mathrm{Cyl}_Z(g) \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner", very near end] & D \wedge_Z I_Z^+ \arrow[dl, dashed, "\exists !"] \\
X . &
\end{tikzcd}$$

Hence $g : C \to D$ is a cofibration.

(iii) Given a cofibration $f : A \to B$ in $\mathbf{C}_Z$, consider

$$\begin{tikzcd} Z \arrow[r, "i_A"] \arrow[d, "i_C"'] & A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \vee_Z C \arrow[r, "f \vee_Z id_C"'] & B \vee_Z C. \end{tikzcd}$$

By the pasting law for pushouts, since the left hand square is the pushout defining $A \vee_Z C$ and the outer rectangle is the pushout defining $B \vee_Z C$, we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A \vee_Z C \arrow[r, "f \vee_Z id_C"'] & B \vee_Z C. \end{tikzcd}$$

By (ii), cofibrations are closed under pushouts. Hence the morphism

$$f \vee_Z id_C : A \vee_Z C \longrightarrow B \vee_Z C$$

is a cofibration.

(iv) Given a cofibration $f : A \to B$, by Lemma 3.2.21, there exist morphisms

$$\mathrm{Cyl}_Z(f) \xrightarrow{\quad i \quad} B \wedge_Z I_Z^+ \xrightarrow{\quad r \quad} \mathrm{Cyl}_Z(f)$$

such that $r \circ i = id$, i.e. $r : B \wedge_Z I_Z^+ \longrightarrow (A \wedge_Z I_Z^+) \vee_f B = \mathrm{Cyl}_Z(f)$ is a retraction. Then, for any $C \in \mathbf{C}_Z$, we have

$$((A \wedge_Z C) \wedge_Z I_Z^+) \underset{f \wedge_Z id_C}{\vee} (B \wedge_Z C) \xrightarrow{i \wedge_Z id_C} (B \wedge_Z C) \wedge_Z I_Z^+ \xrightarrow{r \wedge_Z id_C} ((A \wedge_Z C) \wedge_Z I_Z^+) \underset{f \wedge_Z id_C}{\vee} (B \wedge_Z C) \tag{3.2.3}$$

where the morphism $r \wedge_Z id_C$ is a retraction by the properties of the smash product (Lemma 3.2.6 and Corollary 3.2.7). Therefore, by Lemma 3.2.21,

$$r \wedge_Z id_C : A \wedge_Z C \longrightarrow B \wedge_Z C$$

is a cofibration.

□

Further properties of homotopies in $\mathbf{C}_Z$ require the morphism $c_Z : S_Z^0 \to I_Z$ to be cofibration, **P.4'**, and the existence of a transposition map $\tau : I_Z \to I_Z$ in $\mathbf{C}_Z$ as in **P.5'**. Since $S^0 \to I$ is a cofibration in $\mathbf{C}$ by **P.4**, so is $S_Z^0 \longrightarrow I_Z$ (simply take products with $Z$). The next result verifies that $S_Z^0 \longrightarrow I_Z$ is a based cofibration in $\mathbf{C}_Z$.

**3.2.23 Lemma.** *The morphism $c_Z : S^0_Z \longrightarrow I_Z$ is a (based) cofibration in $\mathbf{C}_Z$.*

*Proof.* We need to prove that

$$\mathrm{Cyl}_Z(S^0_Z \to I_Z) \longrightarrow I_Z \wedge_Z I_Z^+$$

has the homotopy extension property, where $\mathrm{Cyl}_Z(S^0_Z \to I_Z) = (S^0_Z \wedge_Z I_Z^+) \vee_Z I_Z$. Consider the following diagram:

$$\begin{tikzcd}
(S^0_Z \times_Z I_Z) \vee_Z I_Z \arrow[r] \arrow[d] & I_Z \times_Z I_Z \arrow[d] \arrow[ddl, dashed] \\
(S^0_Z \wedge_Z I_Z^+) \vee_Z I_Z \arrow[r] \arrow[d] & I_Z \wedge_Z I_Z^+ \\
X. &
\end{tikzcd}$$

where the upper vertical arrows are induced from the morphisms $I_Z \to I_Z^+$ and $\_ \times_Z I_Z^+ \longrightarrow \_ \wedge_Z I_Z^+$. Since $S^0 \to I$ is an unbased cofibration, we have the dotted extension. However, since the morphism $(S^0_Z \times_Z I) \vee_Z I \to X$ factors through $(S^0_Z \wedge_Z I_Z^+) \vee_Z I_Z$, we deduce that the dotted morphism is actually a based homotopy, i.e. it factors through $I_Z \wedge_Z I_Z^+$ as required. $\square$

We wish to prove that $X \to \mathrm{Cyl}_Z(f)$ is a cofibration. In order to do so, the following result is needed.

**3.2.24 Lemma.** *For any $X \in \mathbf{C}_Z$, the morphism*

$$X \wedge_Z (S^0_Z)^+ \to X \wedge_Z I_Z^+,$$

*where $X \wedge_Z (S^0_Z)^+ \cong X \vee_Z X$, is a cofibration in $\mathbf{C}_Z$.*

*Proof.* Since $c_Z : S^0_Z \to I_Z$ is a cofibration in $\mathbf{C}_Z$ by Lemma 3.2.23, so is

$$c_Z^+ : (S^0_Z)^+ \to I_Z^+.$$

since cofibrations are preserved under pushouts by Lemma 3.2.22, (ii). Then, since the same Lemma tells us in (iv) that the smash product preserves cofibrations, the morphism

$$X \wedge_Z (S^0_Z)^+ \xrightarrow{id_X \wedge_Z c_Z^+} X \wedge_Z I_Z^+$$

is also a cofibration where $X \wedge_Z (S^0_Z)^+ \cong X \wedge_Z (Z \vee_\emptyset Z) \cong X \vee_Z X$. $\square$

**3.2.25 Lemma.** *For any morphism $f : X \to Y$ in $\mathbf{C}_Z$, the standard morphism $\sigma : X \to \mathrm{Cyl}_Z(f)$ is a cofibration in $\mathbf{C}_Z$.*

*Proof.* Suppose we are given a homotopy extension problem along $\sigma$ consisting of compatible morphisms

$$h : X \wedge_Z I_Z^+ \to W$$

and

$$k : \mathrm{Cyl}_Z(f) \to W.$$

By Lemma 3.2.24 above, we can solve the corresponding homotopy extension problem along

$$X \vee_Z X \cong X \wedge_Z (S_Z^0)^+ \to X \wedge_Z I_Z^+,$$

given by

$$h \vee_Z l : (X \vee_Z X) \wedge_Z I_Z^+ \to W$$

and

$$k|_{X \wedge_Z I_Z^+} : X \wedge_Z I_Z^+ \to W,$$

where the morphism $l$ is the composite

$$X \wedge_Z I_Z^+ \to X \xrightarrow{f} Y \xrightarrow{k|_Y} W$$

(so is constant as a homotopy). Then the solution of this latter problem is a homotopy

$$H : (X \wedge_Z I_Z^+) \wedge_Z I_Z^+ \to W.$$

The dotted morphism in the following pushout diagram then solves the original homotopy extension problem:

$$\begin{tikzcd}
X \wedge_Z I_Z^+ \arrow[r, "f \wedge_Z I_Z^+"] \arrow[dd] & Y \wedge_Z I_Z^+ \arrow[dd] \arrow[dr, bend left] & \\
 & & Y \arrow[dd, "k|_Y"] \\
(X \wedge_Z I_Z^+) \wedge_Z I_Z^+ \arrow[r] \arrow[drr, bend right, "H"'] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Cyl}_Z(f) \wedge_Z I_Z^+ \arrow[dr, dashed] & \\
 & & W
\end{tikzcd}$$

We note that here we have used the fact that $\_ \wedge_Z I_Z^+$ preserves pushouts (by Lemma 3.2.6, (v)) in order to identify $\mathrm{Cyl}_Z(f \wedge_Z I_Z^+) \cong \mathrm{Cyl}_Z(f) \wedge_Z I_Z^+$. Hence $\sigma : X \to \mathrm{Cyl}_Z(f)$ is a cofibration as claimed. $\square$

**3.2.26 Lemma.** *For any $X \in \mathbf{C}_Z$, the morphism $\kappa : X \to \mathrm{C}_z(X)$ is a cofibration in $\mathbf{C}_Z$.*

*Proof.* Consider $c_Z : S^0_z \to I_z$ which is a cofibration by Lemma 3.2.23. By Lemma 3.2.22, (ii), cofibrations are preserved by the smash product. Hence

$$id_X \wedge_z c_Z : X \wedge_z S^0_z \longrightarrow X \wedge_z I_z$$

is also a cofibration. Since $X \wedge_z S^0_z \cong X$ by Lemma 3.2.6, (iii), and

$$X \wedge_z I_z = \mathrm{C}_z(X),$$

the morphism $X \to \mathrm{C}_z(X)$ is a cofibration, as required. □

**3.2.27 Lemma.** *Given cofibrations $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{C}_Z$, the induced morphism $\epsilon : B/A \to C/A$ is a cofibration and*

$$(C/A)/(B/A) \cong C/B.$$

*Proof.* Assume $f : A \to B$ and $g : B \to C$ are cofibrations in $\mathbf{C}_Z$. Consider the diagram below

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p_A"'] & B \arrow[r, "g"] \arrow[d] & C \arrow[d] \\
Z \arrow[r] & B/A \arrow[r, "\epsilon"] \arrow[d] & C/A \arrow[d] \\
 & Z \arrow[r] & (C/A)/(B/A).
\end{tikzcd} \tag{3.2.4}$$

The top left hand square is the pushout defining $B/A$ and the top horizontal rectangle is the pushout defining $C/A$. Thus by the pasting law for pushouts, the top right hand square is also a pushout. Since $g$ is a cofibration, by Lemma 3.2.22, (ii), the induced morphism

$$\epsilon : B/A \longrightarrow C/A$$

is a cofibration. Then, since the top right hand square is a pushout and the right hand vertical rectangle is the pushout defining $(C/A)/(B/A)$, again by the pasting law the bottom right hand square is a pushout. Hence $(C/A)/(B/A) \cong C/B$ in $\mathbf{C}_Z$.

□

**3.2.28 Lemma.** *The interval object $I_z$ is contractible in $\mathbf{C}_Z$, i.e. $I_z \simeq Z$.*

*Proof.* This follows from the contractibility of $I_Z$ in the ambient category $\mathbf{C}$ (see Lemma 3.1.17) by taking products with $Z$ and noting that the homotopy $I \times I \to I$ preserves $0 \in I$, hence preserves the base-object after taking the product $\_ \times Z$. $\square$

We now define the double interval object consisting of a copy of $I_Z$ glued at its 1-end to the 0-end of another copy of $I_Z$. This will be used in the lemma which comes after and which is required in order to prove that homotopy is an equivalence relation.

**3.2.29 Definition.** The **double interval** object in $\mathbf{C}_Z$ is $I'_Z = I_Z \underset{1=0}{\vee} I_Z$, i.e.

$$\begin{tikzcd} Z \arrow[r, "i_0"] \arrow[d, "i_1"'] & I_Z \arrow[d, "\beta"] \\ I_Z \arrow[r, "\alpha"'] & I'_Z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{3.2.5}$$

where $\beta(0) = \alpha(1) = 1 \in I'_Z$. Write $0 = \alpha(0) \in I'_Z$ and $2 = \beta(1) \in I'_Z$.

**3.2.30 Lemma.** *There exists a morphism $\delta : I_Z \to I'_Z$ in $\mathbf{C}_Z$ such that $\delta(0) = 0$ and $\delta(1) = 2$ and a homotopy $\eta : I_Z \wedge_Z I_Z^+ \to I'_Z$ in $\mathbf{C}_Z$ from $\delta$ to $\alpha : I_Z \to I'_Z$ (the inclusion of the first summand) relative to* $0$.

*Proof.* The morphism $\delta : I_Z \to I'_Z$ in $\mathbf{C}_Z$ is obtained from the analogue in $\mathbf{C}$ (Lemma 3.1.19) by taking the product with $Z$ (which preserves pushouts by **P.2**, so that $I' \times Z \cong I'_Z$). The required homotopy also arises from the analogue in $\mathbf{C}$ by taking the product with $Z$ and noting that the homotopy in $\mathbf{C}$ was constructed to preserve $0 \in I$, and hence preserves the base-object after taking product with $Z$. $\square$

**3.2.31 Lemma.** *Homotopy is an equivalence relation in $\mathbf{C}_Z$.*

*Proof.* Reflexivity: consider the following diagram obtained from **P.4'** as in Remark 3.2.15:

$$\begin{tikzcd} X \vee_Z X \arrow[rr, "id_X \vee_Z id_X"] \arrow[dr, "\bar{c_Z}"'] & & X. \\ & X \wedge_Z I_Z^+ \arrow[ur, "h"'] & \end{tikzcd}$$

Then, given $f : X \to Y$ in $\mathbf{C}_Z$, we have:

$$\begin{tikzcd} X \vee_Z X \arrow[rr, "id_X \vee_Z id_X"] \arrow[dr, "\bar{c_Z}"'] & & X \arrow[r, "f"] & Y. \\ & X \wedge_Z I_Z^+ \arrow[ur] \arrow[urr, bend right, "h"'] & & \end{tikzcd}$$

This gives a (based) homotopy $h : X \wedge_Z I^+_Z \to Y$ from $f$ to $f$ in $\mathbf{C}_Z$. To prove symmetry: let $h$ be a (based) homotopy from $f : X \to Y$ to $g : X \to Y$, i.e.

$$\begin{tikzcd}
X \vee_Z X \arrow[rr, "f \vee_Z g"] \arrow[dr, "\bar{c_Z}"'] & & Y. \\
& X \wedge_Z I^+_Z \arrow[ur, "h"'] &
\end{tikzcd}$$

By Remark 3.1.2, (i), there exists a transposition morphism $\tau : X \vee_Z X \to X \vee_Z X$. Thus we have

$$\begin{tikzcd}
X \vee_Z X \arrow[r, "\tau"] \arrow[rrr, bend left, "g \vee_Z f"] & X \vee_Z X \arrow[rr, "f \vee_Z g"] \arrow[dr, "\bar{c_Z}"'] & & Y., \\
& & X \wedge_Z I^+_Z \arrow[ur, "h"'] &
\end{tikzcd}$$

which gives the required homotopy from $g$ to $f$. To prove transitivity: let $\epsilon$ be a (based) homotopy between maps $f, g : X \to Y$, i.e.

$$\begin{tikzcd}
X \vee_Z X \arrow[rr, "f \vee_Z g"] \arrow[dr, "\bar{c_Z}"'] & & Y, \\
& X \wedge_Z I^+_Z \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

and $\zeta$ another (based) homotopy between maps $g, h : X \to Y$, i.e.

$$\begin{tikzcd}
X \vee_Z X \arrow[rr, "g \vee_Z h"] \arrow[dr, "\bar{c_Z}"'] & & Y. \\
& X \wedge_Z I^+_Z \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

Consider the double interval object $I'_Z$, defined via the pushout square (3.2.5). Then $(I'_Z)^+$ is defined via the pushout below:

$$\begin{tikzcd}
Z \vee_\emptyset Z = S^0_Z \arrow[r, "\upsilon_0"] \arrow[d, "\upsilon_1"'] & I^+_Z = I_Z \vee_\emptyset Z \arrow[d, "\tilde{\beta}"] \\
I_Z \vee_\emptyset Z = I^+_Z \arrow[r, "\tilde{\alpha}"'] \arrow[ur, phantom, "\ulcorner"] & (I'_Z)^+
\end{tikzcd}$$

where $\upsilon_0$ and $\upsilon_1$ map the base-object of $S^0_Z$ to the disjoint base-object in $I^+_Z$, and the remaining copy of $Z$ in $S^0_Z$ to the 0 and 1 ends of the interval in $I^+_Z$, respectively. Then, in order to glue homotopies $\epsilon$ and $\zeta$ together, consider the following pushout obtained from the above diagram by taking the smash product with $X$ (by Lemma 3.2.6, (v)):

$$\begin{tikzcd}
X \wedge_Z S^0_Z \arrow[r, "id_X \wedge_Z v_0"] \arrow[d, "id_X \wedge_Z v_1"'] & X \wedge_Z I^+_Z \arrow[d, "id_X \wedge_Z \tilde{\beta}"] \arrow[ddr, bend left, "\zeta"] & \\
X \wedge_Z I^+_Z \arrow[r, "id_X \wedge_Z \tilde{\alpha}"'] \arrow[drr, bend right, "\epsilon"'] & X \wedge_Z (I'_Z)^+ \arrow[dr, dashed, "\exists! \gamma"] & \\
& & Y,
\end{tikzcd}$$

where $X \wedge_Z S^0_Z \cong X$ by Lemma 3.2.6, (iii). From the homotopies $\epsilon : X \wedge_Z I^+_Z \to Y$ between $f$ and $g$, and $\zeta : X \wedge_Z I^+_Z \to Y$ between $g$ and $h$, and the universal property of the pushout, there exists a unique morphism $\gamma : X \wedge_Z (I'_Z)^+ \to Y$ which is a (based) homotopy between $f$ and $h$, indexed by $I'_Z$. By Lemma 3.1.19 there exists a morphism $\delta : I \to I'$ in $\mathbf{C}$. The morphism $\delta_Z : I_Z \to I'_Z$ in $\mathbf{C}_Z$ is obtained by taking the product with $Z$ (by **P.2**). Thus a family of maps indexed by $I'_Z$ can be replaced by a family indexed by $I_Z$. It follows that $\gamma : X \wedge_Z I^+_Z \to Y$ is the required (based) homotopy between maps $f$ and $h$. □

**3.2.32 Lemma.** *Given $f : X \to Y$ in $\mathbf{C}_Z$, the morphism $i : Y \to \mathrm{Cyl_Z}(f)$ is a homotopy equivalence.*

*Proof.* By definition, the morphism $i : Y \to \mathrm{Cyl_Z}(f)$ is a pushout:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dd, bend right=60, "id"'] & Y \arrow[d, "i"] \arrow[ddr, bend left, "id"] & \\
\mathrm{Cyl_Z}(X) \arrow[r, "j"'] \arrow[d] & \mathrm{Cyl_Z}(f) \arrow[dr, dashed, "\exists! r"] & \\
X \arrow[rr, "f"'] & & Y.
\end{tikzcd}$$

By construction we have $r \circ i = id_Y$. Thus it remains to prove that

$$i \circ r \simeq id_{\mathrm{Cyl_Z}(f)}.$$

To do this we *glue* the homotopy

$$h : \mathrm{Cyl_Z}(X) \wedge_Z I^+_Z \to \mathrm{Cyl_Z}(X)$$

from the identity to the morphism $\mathrm{Cyl_Z}(X) \to \mathrm{Cyl_Z}(X)$ given by $(x, t) \mapsto (x, 1)$ collapsing the cylinder on to one end to the constant homotopy $Y \wedge_Z I^+_Z \to Y$ at $id_Y$.

The square in the diagram below is used for the required gluing, and by Lemma 3.2.6, (v), it is a pushout square:

$$\begin{tikzcd}
X \wedge_Z I_Z^+ \arrow[r, "f \wedge_Z I_Z^+"] \arrow[d] & Y \wedge_Z I_Z^+ \arrow[r] \arrow[d] & Y \arrow[dd, "i"] \\
\mathrm{Cyl}_Z(X) \wedge_Z I_Z^+ \arrow[r] \arrow[d, "h"'] & \mathrm{Cyl}_Z(f) \wedge_Z I_Z^+ \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\exists! k"] & \\
\mathrm{Cyl}_Z(X) \arrow[rr, "j"'] & & \mathrm{Cyl}_Z(f).
\end{tikzcd}$$

In particular the dotted morphism exists because $h$ is a homotopy relative to the end $X \wedge_Z 1$ of $\mathrm{Cyl}_Z(X)$. Then, by construction the morphism

$$k : \mathrm{Cyl}_Z(f) \wedge_Z I_Z^+ \to \mathrm{Cyl}_Z(f)$$

is homotopy from the identity to $i \circ r$.

□

**3.2.33 Definition.** The **set of (based) homotopy classes of maps from** $X$ **to** $Y$ in $\mathbf{C}_Z$ is defined to be

$$[X, Y] = \mathbf{C}_Z(X, Y)/\sim$$

where $\sim$ is the homotopy equivalence relation on the set of morphisms from $X$ to $Y$, i.e.

$$[X, Y] = \{[f]_\sim \mid f \in \mathbf{C}_Z(X, Y)\}$$

with $[f]_\sim = \{g \in \mathbf{C}_Z(X, Y) \mid g \sim f\}$.

**3.2.34 Definition.** An object $X \in \mathbf{C}_Z$ is an **H-cogroup** if there is a morphism

$$X \xrightarrow{\delta} X \vee_Z X,$$

(comultiplication), and a coinverse

$$X \xrightarrow{\tau} X$$

such that

(i) $\delta$ is H-counital: $X \xrightarrow{\delta} X \vee_Z X \xrightarrow{<id,e>} X$ and $X \xrightarrow{\delta} X \vee_Z X \xrightarrow{<e,id>} X$ are homotopic to the identity (the morphism $e : X \to X$ is given by $e = i_X p_X$ where $i_X : Z \to X$ and $p_X : X \to Z$, i.e. $e : X \to X$ is the constant morphism to the base-object).

(ii) $\delta$ is H-coassociative: the diagram

$$\begin{tikzcd} X \arrow[r, "\delta"] \arrow[d, "\delta"'] & X \vee_Z X \arrow[d, "\delta \vee_Z id"] \\ X \vee_Z X \arrow[r, "id \vee_Z \delta"'] & X \vee_Z X \vee_Z X \end{tikzcd}$$

commutes up to homotopy.

(iii) $\tau$ is H-coinverse for $\delta$: the composites

$$X \xrightarrow{\delta} X \vee_Z X \xrightarrow{<\tau, id>} X$$

and

$$X \xrightarrow{\delta} X \vee_Z X \xrightarrow{<id, \tau>} X$$

are homotopic to $e : X \to X$.

**3.2.35 Proposition.** *$S^1_Z \in \mathbf{C}_Z$ is an H-cogroup object.*

*Proof.* This follows provided the pointed 1-sphere $S^1_*$ is an H-cogroup object in the category $\mathbf{C}_*$, by taking products with $Z$ throughout, and using the fact that $\_ \times_Z Z$ distributes overs pushouts. The H-cogroup structure of $S^1_*$ is induced from the interval doubling map

$$I_* \xrightarrow{\delta} I_* \vee_* I_*$$

and the transposition map

$$I_* \xrightarrow{\tau} I_*.$$

The homotopy conunitality and coassociativity follow from the fact that $\delta$ is homotopic to the inclusion of the first (or second) interval in $I_* \vee_* I_*$. The fact that $\tau$ is an H-coinverse follows from the contractibility of the interval. $\square$

**3.2.36 Corollary.** *For any $X, Y \in \mathbf{C}_Z$, the set of homotopy classes $[\Sigma_Z(X), Y]$ is a group, and $[\Sigma^2_Z(X), Y]$ is an abelian group.*

*Proof.* This follows immediately from the facts that

$$\Sigma_Z(X) = S^1_Z \wedge_Z X,$$

that $S^1_Z$ is an H-cogroup object, and that $\_ \wedge_Z X$ preserves pushouts. The proof is the same as in the classical case; see for example [AGP08, Ch.2]. $\square$

## 3.3 Coexact mapping cone sequences in $\mathbf{C}_Z$

**3.3.1 Lemma.** *Let $f : X \to Y$ be a morphism in $\mathbf{C}_Z$ and $f_1 : Y \to \mathrm{C_z}(f)$ be the canonical inclusion. Then the sequence*

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C_z}(f)$$

*is coexact.*

*Proof.* We need to show that for any $W \in \mathrm{Ob}(\mathbf{C}_Z)$, the sequence of homotopy classes

$$[\mathrm{C_z}(f), W]_z \xrightarrow{f_1^*} [Y, W]_z \xrightarrow{f^*} [X, W]_z$$

is exact, i.e.

$$\mathrm{im}(f_1^*) = \ker(f^*) = \{[\phi] \in [Y, W]_z \mid f^*[\phi] = [\phi \circ f] = [e_0]\}$$

where $e_0 : X \to W$ is the constant morphism from $X$ to the base-object in $W$.

We first prove that $\mathrm{im}(f_1^*) \subset \ker(f^*)$. To do this, we show that given $g : \mathrm{C_z}(f) \to W$, the map $g \circ f_1 \circ f$ is nullhomotopic. We have the following diagram

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[r, "f"] \arrow[d, "m"'] & Y \arrow[d, "f_1"] & \\
X \wedge_z I_z^+ \arrow[r, "k"] & \mathrm{C_z}(X) \arrow[r, "n"'] \arrow[ur, phantom, "\ulcorner"] & \mathrm{C_z}(f) \arrow[dr, "g"] & \\
 & & & W
\end{tikzcd}$$

where $g \circ f_1 \circ f = g \circ n \circ m$. The composition $g \circ n \circ k$ is a homotopy from $g \circ n \circ m$ to the constant morphism $e_0$ since it factors through $k : X \wedge_z I_z^+ \to \mathrm{C_z}(X)$. In other words $g \circ f_1 \circ f = g \circ n \circ m$ is nullhomotopic, as required.

We now need to prove that $\ker(f^*) \subset \mathrm{im}(f_1^*)$. Given $u : Y \to W$ such that the composition $u \circ f$ is nullhomotopic, we show that the morphism $u$ extends to $\mathrm{C_z}(f)$.

Let $h$ be the nullhomotopy between $u \circ f$ and the constant morphism $e_0$. By the universal property of the pushout square defining the mapping cylinder on $f$ in $\mathbf{C}_Z$, there exists a unique morphism $\tilde{u} : \mathrm{Cyl_z}(f) \to W$ :

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left, "u"] & \\
X \wedge_Z I_Z^+ \arrow[r] \arrow[ur, phantom, "\ulcorner"] \arrow[drr, bend right, "h"'] & \mathrm{Cyl}_Z(f) \arrow[dr, dashed, "\exists ! \tilde{u}"] & \\
& & W
\end{tikzcd}$$

There is an obvious morphism $I_Z^+ \to I_Z$, and hence also a morphism

$$X \wedge_Z I_Z^+ \to X \wedge_Z I_Z$$

obtained by smashing with $X$. Since $h$ is a nullhomotopy, by definition it must factorise through $X \wedge_Z I_Z$. Thus there exists a unique morphism $\hat{u} : \mathrm{C}_Z(f) \to W$ by the universal property of pushouts:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \arrow[dddr, bend left, "u"] & \\
X \wedge_Z I_Z^+ \arrow[r] \arrow[ur, phantom, "\ulcorner"] \arrow[d] & \mathrm{Cyl}_Z(f) \arrow[d] & \\
X \wedge_Z I_Z \arrow[r] \arrow[ur, phantom, "\ulcorner"] \arrow[ddrr, bend right, dashed] & \mathrm{C}_Z(f) \arrow[ddr, dashed, "\exists ! \hat{u}"'] & \\
& & \\
& & W
\end{tikzcd}$$

Therefore the sequence $X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C}_Z(f)$ is coexact. $\square$

The mapping cone construction,

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\mathbf{f}"] \arrow[d, "id_X \times i_1"'] & \mathbf{Y} \arrow[d, "\mathbf{f_1}"] \\
\mathrm{C}_Z(X) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathbf{C_Z(f)},
\end{tikzcd}$$

can be used to extend any map $f : X \to Y$ in $\mathbf{C}_Z$ to an infinite *coexact sequence* by applying the same construction, in turn, to each map in the sequence. This yields the coexact mapping cone sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C}_Z(f) \xrightarrow{f_2} \mathrm{C}_Z(f_1) \xrightarrow{f_3} \mathrm{C}_Z(f_2) \xrightarrow{f_4} \mathrm{C}_Z(f_3) \xrightarrow{f_5} \mathrm{C}_Z(f_4) \xrightarrow{f_6} \mathrm{C}_Z(f_5)... , \tag{3.3.1}$$

where the $f_k$ are the canonical inclusions. We will now prove that these iterated mapping cones can be replaced by more familiar homotopy equivalent spaces. We follow the standard proof for topological spaces as in [AGP08, p.72-77], but abstracting all the results.

**3.3.2 Lemma.** *Let $M' \xrightarrow{l} M$ be a cofibration in $\mathbf{C}_Z$ and suppose that there exists a homotopy*

$$H : M \wedge_Z I_Z^+ \to M$$

*such that*

*(i)* $H(\_, 0) = id_M$, *i.e.*

$$\begin{tikzcd} M \arrow[r, "id \wedge_Z 0"] \arrow[dr, "id"'] & M \wedge_Z I_Z^+ \arrow[d, "H"] \\ & M, \end{tikzcd}$$

*commutes where the horizontal morphism is induced from $Z \xrightarrow{i_0} I_Z^+$;*

*(ii) the homotopy $H$ preserves $M'$, i.e. there is a commuting diagram:*

$$\begin{tikzcd} M' \wedge_Z I_Z^+ \arrow[r, dashed, "\exists \tilde{H}"] \arrow[d, "l \wedge_Z id"'] & M' \arrow[d, "l"] \\ M \wedge_Z I_Z^+ \arrow[r, "H"'] & M; \end{tikzcd}$$

*(iii) $H$ contracts $M'$ to the base-object, i.e. the diagram*

$$\begin{tikzcd} M' \arrow[r, "p_{M'}"] \arrow[d, "l \wedge_Z 1"'] & Z \arrow[d, "i_M"] \\ M \wedge_Z I_Z^+ \arrow[r, "H"'] & M \end{tikzcd}$$

*commutes where the left hand vertical morphism is induced from $Z \xrightarrow{i_1} I_Z^+$.*

*Then the identification $q : M \to M/M'$ is a homotopy equivalence.*

*Proof.* On the one hand, using property (iii) of the homotopy $H$, we can define a map

$$s : M/M' \longrightarrow M$$

as follows:

$$\begin{tikzcd} M' \arrow[r, "l"] \arrow[d, "p_{M'}"'] & M \arrow[d, "q"] \arrow[ddr, bend left, "H \circ (id_M \wedge_Z \upsilon_1)"] & \\ Z \arrow[r] \arrow[drr, bend right] \arrow[ur, phantom, "\ulcorner"] & M/M' \arrow[dr, dashed, "\exists ! s"] & \\ & & M. \end{tikzcd}$$

Then by construction, H is a homotopy from $id_M$ to $s \circ q$. On the other hand we can show that $H$ induces a homotopy

$$\bar{H} : (M/M') \wedge_Z I_Z^+ \longrightarrow M/M'$$

such that $\bar{H} \circ (q \wedge_Z I_Z^+) = q \circ H$, i.e.

$$\begin{tikzcd}
M \wedge_Z I_Z^+ \arrow[r, "H"] \arrow[d, "q \wedge_Z id"'] & M \arrow[d, "q"] \\
(M/M') \wedge_Z I_Z^+ \arrow[r, dashed, "\bar{H}"'] & M/M'.
\end{tikzcd}$$

Since taking the smash product with $I_Z^+$ preserves pushouts in $\mathbf{C}_Z$ (by Lemma 3.2.6, (v)), we can consider the following diagram:

$$\begin{tikzcd}
M' \wedge_Z I_Z^+ \arrow[r] \arrow[d] & M \wedge_Z I_Z^+ \arrow[r, "H"] \arrow[d, "q \wedge_Z id"] & M \arrow[dd, "q"] \\
Z \wedge_Z I_Z^+ \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & (M/M') \wedge_Z I_Z^+ \arrow[dr, dashed, "\exists ! \bar{H}"] & \\
Z \arrow[rr] & & M/M'
\end{tikzcd}$$

where the outer square commutes by property (ii). Thus, by the universal property of the pushout, there exists a unique morphism $\bar{H} : (M/M') \wedge_Z I_Z^+ \longrightarrow M/M'$ making the diagram commute. Then considering

$$\begin{tikzcd}
M \arrow[r, "id \wedge_Z 0"] \arrow[d, "q"'] & M \wedge_Z I_Z^+ \arrow[r, "H"] \arrow[d, "q \wedge_Z id"] & M \arrow[d, "q"] \\
M/M' \arrow[r, "id \wedge_Z 0"'] & (M/M') \wedge_Z I_Z^+ \arrow[r, "\bar{H}"'] & M/M',
\end{tikzcd}$$

we have $\bar{H}(q(\_), 0) = q(\_)$. From the following square and the universal property of pushouts

$$\begin{tikzcd}
M' \arrow[r, "l"] \arrow[d, "p_{M'}"'] & M \arrow[d, "q"] \arrow[ddr, bend left, "q"] & \\
Z \arrow[r] \arrow[drr, bend right, "i_{M/M'}"'] \arrow[ur, phantom, "\ulcorner"] & M/M' \arrow[dr, "\bar{H}(_,0)"'] & \\
& & M/M',
\end{tikzcd}$$

we have $\bar{H}(\_, 0) = id_{M/M'}$. Then, from the following diagram,

$$\begin{tikzcd}
M \arrow[r, "id\wedge_Z 1"] \arrow[d, "q"'] \arrow[rr, bend left, "s\circ q"] & M \wedge_Z I_Z^+ \arrow[r, "H"] \arrow[d, "q\wedge_Z id"] & M \arrow[d, "q"] \\
M/M' \arrow[r, "id\wedge_Z 1"'] & (M/M') \wedge_Z I_Z^+ \arrow[r, "\bar{H}"'] & M/M',
\end{tikzcd}$$

we see that $\bar{H}(q(\_),1) = q \circ s \circ q$. Hence $\bar{H}(\_,1) = q \circ s$.

$\square$

Given $X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C_z}(f)$, let $j_1 : \mathrm{C_z}(Y) \longrightarrow \mathrm{C_z}(f_1)$ be the morphism arising in the definition of the mapping cone of $f_1$, as below:

$$\begin{tikzcd}
Y \arrow[r, "f_1"] \arrow[d] & \mathrm{C_z}(f) \arrow[d] \\
\mathrm{C_z}(Y) \arrow[r, "j_1"'] \arrow[ur, phantom, "\ulcorner"] & \mathrm{C_z}(f_1).
\end{tikzcd}$$

**3.3.3 Lemma.** *Given $f : X \to Y$ in $\mathbf{C}_Z$, consider the canonical morphism $f_1 : Y \to \mathrm{C_z}(f)$. There exists a homotopy equivalence*

$$\mathrm{C_z}(f_1) \longrightarrow \mathrm{C_z}(f_1)/\,\mathrm{C_z}(Y).$$

*Proof.* To prove the existence of the homotopy equivalence

$$\mathrm{C_z}(f_1) \longrightarrow \mathrm{C_z}(f_1)/\,\mathrm{C_z}(Y)$$

consider Lemma 3.3.2. Let $M' = \mathrm{C_z}(Y)$ and $M = \mathrm{C_z}(f_1)$, and

$$j_1 : \mathrm{C_z}(Y) \to \mathrm{C_z}(f_1).$$

Then we prove there exists a homotopy $\mathrm{C_z}(f_1) \wedge_Z I_Z^+ \longrightarrow \mathrm{C_z}(f_1)$ with the required properties (i)-(iii).

By Lemma 3.2.26 we know that $\kappa : X \to \mathrm{C_z}(X)$ is a cofibration in $\mathbf{C}_Z$. Since the mapping cone of $f$ is defined via the pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\kappa"'] & Y \arrow[d, "f_1"] \\
\mathrm{C_z}(X) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{C_z}(f),
\end{tikzcd}$$

the morphism $f_1 : Y \to \mathrm{C_z}(f)$ is a cofibration by Lemma 3.2.22, (ii). Considering the pushout

$$\begin{tikzcd}
Y \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \mathrm{C}_{\mathrm{Z}}(f) \arrow[d] \\
\mathrm{C}_{\mathrm{Z}}(Y) \arrow[r, "j_1"'] & \mathrm{C}_{\mathrm{Z}}(f_1),
\end{tikzcd}$$

we deduce that $j_1 : \mathrm{C}_{\mathrm{Z}}(Y) \to \mathrm{C}_{\mathrm{Z}}(f_1)$ is also cofibration. Take

$$\mathrm{C}_{\mathrm{Z}}(Y) \wedge_Z I_Z^+ \xrightarrow{\ h\ } \mathrm{C}_{\mathrm{Z}}(Y) \xrightarrow{\ j_1\ } \mathrm{C}_{\mathrm{Z}}(f_1),$$

where $h$ is a homotopy from the identity on $\mathrm{C}_{\mathrm{Z}}(Y)$ to $p : \mathrm{C}_{\mathrm{Z}}(Y) \to Z$. Then, since $j_1$ is a cofibration, there exists a morphism $H$ which completes the following commutative triangle

$$\begin{tikzcd}
(\mathrm{C}_{\mathrm{Z}}(Y) \wedge_Z I_Z^+) \vee_{j_1} \mathrm{C}_{\mathrm{Z}}(f_1) \arrow[r, "i"] \arrow[d, "(j_1 h)\vee_{j_1} id"'] & \mathrm{C}_{\mathrm{Z}}(f_1) \wedge_Z I_Z^+ \arrow[dl, dashed, "\exists H"] \\
\mathrm{C}_{\mathrm{Z}}(f_1). &
\end{tikzcd}$$

The morphism $H$ is a homotopy such that

1. $H(\_, 0) = id_{\mathrm{C}_{\mathrm{Z}}(f_1)}$, i.e.

$$\begin{tikzcd}
\mathrm{C}_{\mathrm{Z}}(f_1) \arrow[r, "id\wedge_Z 0"] \arrow[dr, "id"'] & \mathrm{C}_{\mathrm{Z}}(f_1) \wedge_Z I_Z^+ \arrow[d, "H"] \\
& \mathrm{C}_{\mathrm{Z}}(f_1),
\end{tikzcd}$$

2. the following diagram commutes

$$\begin{tikzcd}
\mathrm{C}_{\mathrm{Z}}(Y) \wedge_Z I_Z^+ \arrow[r, "j_1\wedge_Z id"] \arrow[d, "p"'] & \mathrm{C}_{\mathrm{Z}}(f_1) \wedge_Z I_Z^+ \arrow[r, "H"] & \mathrm{C}_{\mathrm{Z}}(f_1) \arrow[d, "q"] \\
Z \arrow[rr, "i"'] & & \mathrm{C}_{\mathrm{Z}}(f_1)/\,\mathrm{C}_{\mathrm{Z}}(Y),
\end{tikzcd}$$

3. the square below commutes

$$\begin{tikzcd}
\mathrm{C}_{\mathrm{Z}}(Y) \arrow[r, "p"] \arrow[d, "j_1\wedge_Z 1"'] & Z \arrow[d, "i"] \\
\mathrm{C}_{\mathrm{Z}}(f_1) \wedge_Z I_Z^+ \arrow[r, "H"'] & \mathrm{C}_{\mathrm{Z}}(f_1).
\end{tikzcd}$$

Therefore, by Lemma 3.3.2,

$$q : \mathrm{C}_{\mathrm{Z}}(f_1) \to \mathrm{C}_{\mathrm{Z}}(f_1)/\,\mathrm{C}_{\mathrm{Z}}(Y)$$

is a homotopy equivalence. So $\mathrm{C}_{\mathrm{Z}}(f_1) \simeq \mathrm{C}_{\mathrm{Z}}(f_1)/\,\mathrm{C}_{\mathrm{Z}}(Y)$ in $\mathbf{C}_Z$. $\square$

**3.3.4 Lemma.** $\mathrm{C_z}(f_1) \simeq \Sigma_{\mathrm{z}}(X)$.

*Proof.* Consider the following diagram consisting of the pushout square defining $\mathrm{C_z}(f_1)$ and a further pushout square of the maps $j$ and $p_{\mathrm{C_z}(Y)}$:

$$\begin{tikzcd}
Y \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{C_z}(f) \arrow[d] \\
\mathrm{C_z}(Y) \arrow[r, "j"'] \arrow[d, "p_{\mathrm{C_z}(Y)}"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{C_z}(f_1) \arrow[d, "q"] \\
Z \arrow[r] & \mathrm{C_z}(f_1)/\,\mathrm{C_z}(Y).
\end{tikzcd}$$

Since the outside rectangle is also a pushout, we have

$$\mathrm{C_z}(f_1)/\,\mathrm{C_z}(Y) \cong \mathrm{C_z}(f)/Y. \tag{3.3.2}$$

Now consider the diagram below where the left pushout square is the definition of $\mathrm{C_z}(f)$ and the right pushout square is the collapse of $Y$ inside $\mathrm{C_z}(f)$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Z \arrow[d] \\
\mathrm{C_z}(X) \arrow[r] & \mathrm{C_z}(f) \arrow[r] & \mathrm{C_z}(f)/Y.
\end{tikzcd}$$

Since the outside rectangle is the pushout square which defines the suspension of $X$, we have $\mathrm{C_z}(f)/Y \cong \Sigma_{\mathrm{z}}(X)$. By Lemma 3.3.3, we have

$$\mathrm{C_z}(f_1) \simeq \mathrm{C_z}(f_1)/\,\mathrm{C_z}(Y),$$

and from (3.3.2) above, we know that

$$\mathrm{C_z}(f_1)/\,\mathrm{C_z}(Y) \cong \mathrm{C_z}(f)/Y.$$

Hence

$$\mathrm{C_z}(f_1) \simeq \mathrm{C_z}(f_1)/\,\mathrm{C_z}(Y) \cong \mathrm{C_z}(f)/Y \cong \Sigma_{\mathrm{z}}(X).$$

□

The following is a corollary of Lemma 3.3.1 and Lemma 3.3.4.

**3.3.5 Corollary.** *Given a morphism $f : X \to Y$ in $\mathbf{C}_Z$, the* **shortened Puppe sequence**

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C_z}(f) \xrightarrow{f_2} \Sigma_{\mathrm{z}}(X) \tag{3.3.3}$$

*is coexact.*

*Proof.* From Lemma 3.3.1 we know that $X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C_Z}(f)$ is coexact. Applying the same reasoning to the morphism $f_1$ as we did to $f$, we obtain an extended sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C_Z}(f) \xrightarrow{f_2} \mathrm{C_Z}(f_1)$$

which is also coexact (at $Y$ and $\mathrm{C_Z}(f)$). By Lemma 3.3.4, we have

$$\mathrm{C_Z}(f_1) \simeq \Sigma_{\mathrm{Z}}(X),$$

hence the extended sequence is homotopy equivalent to the sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C_Z}(f) \xrightarrow{f_2} \Sigma_{\mathrm{Z}}(X)$$

which is also coexact and is called the shortened Puppe sequence of $f$. □

**3.3.6 Lemma.** *A homotopy commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r] \arrow[d, "v"] & \mathrm{C_Z}(f) \arrow[r] \arrow[d, dashed, "W"] & \Sigma_{\mathrm{Z}}(A) \arrow[d, "\Sigma_{\mathrm{Z}}(u)"] \\ A' \arrow[r, "f'"'] & B' \arrow[r] & \mathrm{C_Z}(f') \arrow[r] & \Sigma_{\mathrm{Z}}(B') \end{tikzcd}$$

*in $\mathbf{C}_Z$ can be completed by the dotted morphism (so that the result homotopy commutes).*

*Proof.* The homotopy $h$ from $f'u$ to $vf$ allows us to construct a morphism

$$h \vee_f v : \mathrm{Cyl_Z}(f) \to B'$$

as the unique dotted morphism completing the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "id\wedge_Z v_1"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \arrow[ddr, bend left, "v"] & \\ A \wedge_Z I_Z^+ \arrow[r] \arrow[drr, bend right, "h"'] & \mathrm{Cyl_Z}(f) \arrow[dr, dashed, "h\vee_f v"] & \\ & & B'. \end{tikzcd}$$

From this and the morphism $\delta : I_Z \to I_Z \vee_Z I_Z$, we can construct morphisms

$$\mathrm{Cyl_Z}(f) \to (A \wedge_Z I_Z^+) \vee_A \mathrm{Cyl_Z}(f) \to \mathrm{Cyl_Z}(f')$$

where the second is induced from

$$u \wedge_Z id : A \wedge_Z I_Z^+ \to A' \wedge_Z I$$

and

$$h \vee_f v : \mathrm{Cyl}_Z(f) \to B'.$$

This fits into a commuting square

$$\begin{tikzcd} A \arrow[r, "id_A \wedge_Z v_0"] \arrow[d, "u"'] & \mathrm{Cyl}_Z(f) \arrow[d] \\ A' \arrow[r, "id_{A'} \wedge_Z v_0"'] & \mathrm{Cyl}_Z(f'). \end{tikzcd}$$

and so induces a morphism $w : \mathrm{C}_Z(f) \to \mathrm{C}_Z(f')$. By construction, this has the required properties. $\square$

**3.3.7 Lemma.** *If $f : A \to B$ is a cofibration in $\mathbf{C}_Z$, then $\mathrm{C}_Z(f) \simeq B/A$.*

*Proof.* Suppose $f : A \to B$ is a cofibration in $\mathbf{C}_Z$. Consider the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & B \arrow[d] \\ \mathrm{C}_Z(A) \arrow[r, "j"'] & \mathrm{C}_Z(f). \end{tikzcd}$$

By Lemma 3.2.22, (ii), since $f$ is a cofibration, so is $j$. We now construct a commutative diagram

$$\begin{tikzcd} \mathrm{C}_Z(A) \wedge_Z I_Z^+ \arrow[r, "H"] \arrow[d] & \mathrm{C}_Z(A) \arrow[d] \\ \mathrm{C}_Z(f) \wedge_Z I_Z^+ \arrow[r, dashed, "G"'] & \mathrm{C}_Z(f) \end{tikzcd}$$

where

$$H : \mathrm{C}_Z(A) \wedge_Z I_Z^+ \longrightarrow \mathrm{C}_Z(A)$$

is a homotopy from the identity on $\mathrm{C}_Z(A)$ to $p : \mathrm{C}_Z(A) \to 0$ (the collapse of $\mathrm{C}_Z(A)$ down to its vertex). Since $j : \mathrm{C}_Z(A) \to \mathrm{C}_Z(f)$ is a cofibration, there exists a morphism $G$ which completes the following diagram:

$$\begin{tikzcd} \mathrm{Cyl}_Z(j) \arrow[r, "i"] \arrow[d, "H \vee_j id"'] & \mathrm{C}_Z(f) \wedge_Z I_Z^+ \arrow[dl, dashed, "\exists G"] \\ \mathrm{C}_Z(f). & \end{tikzcd} \tag{3.3.4}$$

The homotopy $G$ satisfies the properties of Lemma 3.3.2. Therefore

$$q : \mathrm{C}_Z(f) \longrightarrow \mathrm{C}_Z(f)/\,\mathrm{C}_Z(A)$$

is a homotopy equivalence. Finally, consider

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
\mathrm{C}_{\mathrm{z}}(A) \arrow[r, "j"'] \arrow[d] & \mathrm{C}_{\mathrm{z}}(f) \arrow[d, "q"] \\
Z \arrow[r] & B/A.
\end{tikzcd}$$

The top square is the pushout defining $\mathrm{C}_{\mathrm{z}}(f)$, the bottom square is the pushout defining $\mathrm{C}_{\mathrm{z}}(f)/\,\mathrm{C}_{\mathrm{z}}(A)$, and the outer rectangle is the pushout defining $B/A$. So $\mathrm{C}_{\mathrm{z}}(f)/\,\mathrm{C}_{\mathrm{z}}(A) \cong B/A$. Hence we have

$$\mathrm{C}_{\mathrm{z}}(f) \simeq \mathrm{C}_{\mathrm{z}}(f)/\,\mathrm{C}_{\mathrm{z}}(A) \cong B/A.$$

□

**3.3.8 Lemma.** $\mathrm{C}_{\mathrm{z}}(f_2) \simeq \Sigma_{\mathrm{z}}(Y)$.

*Proof.* This follows the same reasoning as the proof of Lemma 3.3.4 using that the identification $q_2 : \mathrm{C}_{\mathrm{z}}(f_2) \to \mathrm{C}_{\mathrm{z}}(f_2)/\,\mathrm{C}_{\mathrm{z}}(\mathrm{C}_{\mathrm{z}}(f))$ is a homotopy equivalence. □

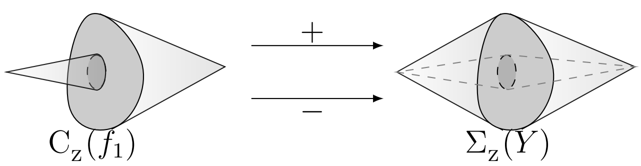
**3.3.9 Lemma.** *Let $q_1 : \mathrm{C}_{\mathrm{z}}(f_1) \to \Sigma_{\mathrm{z}}(X)$ and $q_2 : \mathrm{C}_{\mathrm{z}}(f_2) \to \Sigma_{\mathrm{z}}(Y)$ be the two homotopy equivalences as in Lemma 3.3.4 and Lemma 3.3.8, and consider the transposition map $\tau : I_{\mathrm{z}} \to I_{\mathrm{z}}$ as in **P.5'**. Then the following square commutes up to homotopy:*

$$\begin{tikzcd}
\mathrm{C}_{\mathrm{z}}(f_1) \arrow[r, "f_3"] \arrow[d, "\tau\circ q_1"'] & \mathrm{C}_{\mathrm{z}}(f_2) \arrow[d, "q_2"] \\
\Sigma_{\mathrm{z}}(X) \arrow[r, "\Sigma_{\mathrm{z}}(f)"'] & \Sigma_{\mathrm{z}}(Y).
\end{tikzcd}$$

*Proof.* Firstly, we notice that the object $\mathrm{C}_{\mathrm{z}}(f_2)$ is not relevant in this situation (since $\mathrm{C}_{\mathrm{z}}(f_2)$ consists just of $\mathrm{C}_{\mathrm{z}}(f_1)$ with some additional cone structure, precisely the cone on $\mathrm{C}_{\mathrm{z}}(f)$, which is collapsed down again under the map $q_2$). Thus we want to show that the following diagram commutes up to homotopy:

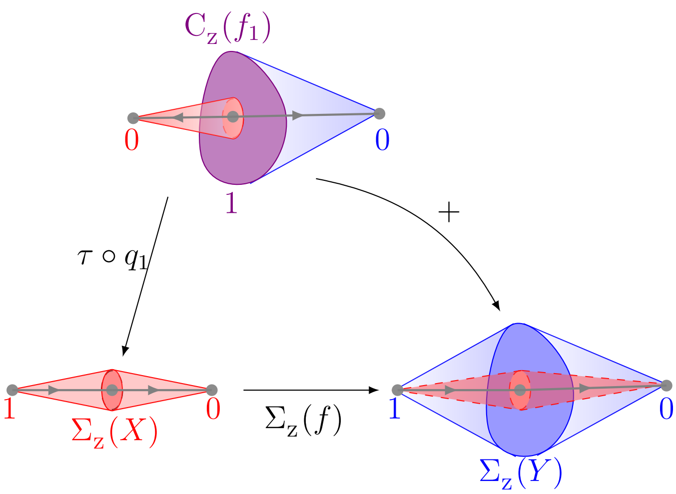
$$\begin{tikzcd}
& \mathrm{C}_{\mathrm{z}}(f_1) \arrow[dl, "\tau\circ q_1"'] \arrow[dr, "+"] & \\
\Sigma_{\mathrm{z}}(X) \arrow[rr, "\Sigma_{\mathrm{z}}(f)"'] & & \Sigma_{\mathrm{z}}(Y).
\end{tikzcd}$$

There exist two maps from $\mathrm{C}_{\mathrm{z}}(f_1)$ to $\Sigma_{\mathrm{z}}(Y)$ which are homotopic up to application of the transposition map $\tau$:



where $+$ is the direct map from $\mathrm{C_z}(f_1)$ to $\Sigma_\mathrm{z}(Y)$ given by collapsing $\mathrm{C_z}(f) \subset \mathrm{C_z}(f_1)$ (the left hand side), and $-$ is the map from $\mathrm{C_z}(f_1)$ to $\Sigma_\mathrm{z}(Y)$ given by collapsing $\mathrm{C_z}(Y) \subset \mathrm{C_z}(f_1)$ (the right hand side) to obtain the suspension on $X$, and then mapping that into $\Sigma_\mathrm{z}(Y)$ via $\Sigma_\mathrm{z}(f)$.

This becomes evident upon careful consideration of the different mapping cone constructions involved. In particular, noting that given a map $g : A \to B$, it is the target of this map, $B$, which is located at the 1-end of the mapping cone $\mathrm{C_z}(g)$. The following diagram illustrates these underlying mapping cone constructions and clarifies that, up to application of $\tau$ to flip the interval direction of $\Sigma_\mathrm{z}(X)$, there certainly exist two maps from $\mathrm{C_z}(f_1)$ to $\Sigma_\mathrm{z}(Y)$ which are homotopic:




□

Thus, up to homotopy, the coexact sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C_z}(f) \xrightarrow{f_2} \mathrm{C_z}(f_1) \xrightarrow{f_3} \mathrm{C_z}(f_2)$$

can be replaced by

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C_z}(f) \xrightarrow{f_2} \Sigma_\mathrm{z}(X) \xrightarrow{\Sigma_\mathrm{z}(f)} \Sigma_\mathrm{z}(Y)$$

which is also coexact. We now prove that this can be extended further by applying the whole process again, but to the map $\Sigma_\mathrm{z} f : \Sigma_\mathrm{z}(X) \to \Sigma_\mathrm{z}(Y)$ rather than $f$, in order to obtain the sequence:

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C_z}(f) \xrightarrow{f_2} \Sigma_\mathrm{z}(X) \xrightarrow{\Sigma_\mathrm{z}(f)} \Sigma_\mathrm{z}(Y) \xrightarrow{(\Sigma_\mathrm{z}(f))_1} \mathrm{C_z}(\Sigma_\mathrm{z}(f)).$$

**3.3.10 Lemma.** $\mathrm{C_z}(\Sigma_\mathrm{z}(X)) \cong \Sigma_\mathrm{z}(\mathrm{C_z}(X))$.

*Proof.* We recall that $\mathrm{C_z}(X) \cong X \wedge_Z I_Z$ and $\Sigma_\mathrm{z}(X) \cong X \wedge_Z S^1_Z$. From Lemma 3.2.6, (iv), we know that $(X \wedge_Z A) \wedge_Z B \cong (X \wedge_Z B) \wedge_Z A$. Thus, letting $A = S^1_Z$ and $B = I_Z$, we obtain the required isomorphism:

$$\mathrm{C_z}(\Sigma_\mathrm{z}(X)) \cong (X \wedge_Z S^1_Z) \wedge_Z I_Z \cong (X \wedge_Z I_Z) \wedge_Z S^1_Z \cong \Sigma_\mathrm{z}(\mathrm{C_z}(X)).$$

□

**3.3.11 Lemma.** $\mathrm{C_z}(f_3) \simeq \mathrm{C_z}(\Sigma_\mathrm{z}(f)) \cong \Sigma_\mathrm{z}(\mathrm{C_z}(f))$.

*Proof.* Given $f : X \to Y$ in $\mathbf{C}_Z$, the mapping cone on the morphism

$$\Sigma_\mathrm{z}(f) : \Sigma_\mathrm{z}(X) \to \Sigma_\mathrm{z}(Y)$$

is defined via the following pushout:

$$\begin{tikzcd}
\Sigma_\mathrm{z}(X) \arrow[r, "\Sigma_\mathrm{z}(f)"] \arrow[d] & \Sigma_\mathrm{z}(Y) \arrow[d] \\
\mathrm{C_z}(\Sigma_\mathrm{z}(X)) \arrow[r, "j"'] & \mathrm{C_z}(\Sigma_\mathrm{z}(f)).
\end{tikzcd}$$

The suspension of the mapping cone on $f$ is defined via:

$$\begin{tikzcd}
\mathrm{C_z}(f) \arrow[r, "\Sigma_\mathrm{z}(f)"] \arrow[d] & Z \arrow[d] \\
\mathrm{C_z}(\mathrm{C_z}(f)) \arrow[r] & \Sigma_\mathrm{z}(\mathrm{C_z}(f)).
\end{tikzcd}$$

By Lemma 3.3.10, $\mathrm{C_z}(\Sigma_\mathrm{z}(X)) \cong \Sigma_\mathrm{z}(\mathrm{C_z}(X))$. Consider the suspensions of the morphisms $f_1 : \mathrm{C_z}(X) \to \mathrm{C_z}(f)$ and $k : Y \to \mathrm{C_z}(f)$, denoted $\Sigma_\mathrm{z}(f_1)$ and $\Sigma_\mathrm{z}(k)$ respectively. By the universal property of pushouts, there exists a unique morphism $u_1 : \mathrm{C_z}(\Sigma_\mathrm{z}(f)) \to \Sigma_\mathrm{z}(\mathrm{C_z}(f))$ such that the following commutes:

$$\begin{tikzcd}
\Sigma_\mathrm{z}(X) \arrow[r, "\Sigma_\mathrm{z}(f)"] \arrow[d] & \Sigma_\mathrm{z}(Y) \arrow[d] \arrow[ddr, bend left, "\Sigma_\mathrm{z}(f_1)"] & \\
\mathrm{C_z}(\Sigma_\mathrm{z}(X)) \arrow[r] \arrow[dr, "\cong"'] & \mathrm{C_z}(\Sigma_\mathrm{z}(f)) \arrow[dr, dashed, "\exists! u_1"] & \\
 & \Sigma_\mathrm{z}(\mathrm{C_z}(X)) \arrow[r, "\Sigma_\mathrm{z}(k)"'] & \Sigma_\mathrm{z}(\mathrm{C_z}(f)).
\end{tikzcd}$$

Now consider the morphism $j : \mathrm{C_z}(\Sigma_\mathrm{z}(X)) \to \mathrm{C_z}(\Sigma_\mathrm{z}(f))$ (arising in the definition of $\mathrm{C_z}(\Sigma_\mathrm{z}(f))$) and the cone (functor) on the morphism

$$f_2 : \mathrm{C_z}(f) \to \Sigma_\mathrm{z}(X),$$

denoted

$$\mathrm{Cone}(f_2) : \mathrm{C}_{\mathrm{z}}(\mathrm{C}_{\mathrm{z}}(f)) \to \mathrm{C}_{\mathrm{z}}(\Sigma_{\mathrm{z}}(X)).$$

Then, by the universal property of the pushout defining $\Sigma_{\mathrm{z}}(\mathrm{C}_{\mathrm{z}}(f))$, there exists a unique morphism $u_2 : \Sigma_{\mathrm{z}}(\mathrm{C}_{\mathrm{z}}(f)) \to \mathrm{C}_{\mathrm{z}}(\Sigma_{\mathrm{z}}(f))$:

$$\begin{tikzcd}
\mathrm{C}_{\mathrm{z}}(f) \arrow[r, "\Sigma_{\mathrm{z}}(f)"] \arrow[d] & Z \arrow[d] \arrow[ddr, bend left, "i"] & \\
\mathrm{C}_{\mathrm{z}}(\mathrm{C}_{\mathrm{z}}(f)) \arrow[r] \arrow[d, "\mathrm{Cone}(f_2)"'] \arrow[ur, phantom, "\ulcorner"] & \Sigma_{\mathrm{z}}(\mathrm{C}_{\mathrm{z}}(f)) \arrow[dr, dashed, "\exists ! u_2"] & \\
\mathrm{C}_{\mathrm{z}}(\Sigma_{\mathrm{z}}(X)) \arrow[rr, "j"'] & & \mathrm{C}_{\mathrm{z}}(\Sigma_{\mathrm{z}}(f)).
\end{tikzcd}$$

Hence, since the composite $\mathrm{C}_{\mathrm{z}}(\Sigma_{\mathrm{z}}(f)) \xrightarrow{u_1} \Sigma_{\mathrm{z}}(\mathrm{C}_{\mathrm{z}}(f)) \xrightarrow{u_2} \mathrm{C}_{\mathrm{z}}(\Sigma_{\mathrm{z}}(f))$ is the identity, we have

$$\mathrm{C}_{\mathrm{z}}(\Sigma_{\mathrm{z}}(f)) \cong \Sigma_{\mathrm{z}}(\mathrm{C}_{\mathrm{z}}(f)).$$

□

**3.3.12 Corollary.** *Suppose $f : X \to Y$ is a morphism in $\mathbf{C}_Z$. The shortened Puppe sequence*

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C}_{\mathrm{z}}(f) \xrightarrow{f_2} \Sigma_{\mathrm{z}}(X)$$

*can be extended to the infinite mapping cone sequence*

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C}_{\mathrm{z}}(f) \xrightarrow{f_2} \Sigma_{\mathrm{z}}(X) \xrightarrow{\Sigma_{\mathrm{z}}(f)} \Sigma_{\mathrm{z}}(Y) \xrightarrow{\Sigma_{\mathrm{z}}(f_1)} \Sigma_{\mathrm{z}}(\mathrm{C}_{\mathrm{z}}(f)) \xrightarrow{\Sigma_{\mathrm{z}}(f_2)} {\Sigma_{\mathrm{z}}}^2(X) \xrightarrow{{\Sigma_{\mathrm{z}}}^2(f)} {\Sigma_{\mathrm{z}}}^2(Y) ... \tag{3.3.5}$$

*which is also coexact. This is known as the* **coexact Puppe sequence of** *$f$.*

*Proof.* Consider the coexact mapping cone sequence

$$X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C}_{\mathrm{z}}(f) \xrightarrow{f_2} \mathrm{C}_{\mathrm{z}}(f_1) \xrightarrow{f_3} \mathrm{C}_{\mathrm{z}}(f_2) \xrightarrow{f_4} \mathrm{C}_{\mathrm{z}}(f_3) \xrightarrow{f_5} \mathrm{C}_{\mathrm{z}}(f_4) \xrightarrow{f_6} \mathrm{C}_{\mathrm{z}}(f_5) ... \,. \tag{3.3.6}$$

We know that $\mathrm{C}_{\mathrm{z}}(f_1) \simeq \Sigma_{\mathrm{z}}(X)$ by Lemma 3.3.4, that $\mathrm{C}_{\mathrm{z}}(f_2) \simeq \Sigma_{\mathrm{z}}(Y)$ by Lemma 3.3.8, and that $f_3 : \mathrm{C}_{\mathrm{z}}(f_1) \to \mathrm{C}_{\mathrm{z}}(f_2)$ is homotopic to $\Sigma_{\mathrm{z}}(f) : \Sigma_{\mathrm{z}}(X) \to \Sigma_{\mathrm{z}}(Y)$ by Lemma 3.3.9. So we have $\mathrm{C}_{\mathrm{z}}(f_3) \simeq \mathrm{C}_{\mathrm{z}}(\Sigma_{\mathrm{z}}(f))$. We can adopt the same notation as for the first part of the sequence, now starting with $\Sigma_{\mathrm{z}}(f)$, so the morphism $f_4$ can be written as $(\Sigma_{\mathrm{z}}(f))_1 : \Sigma_{\mathrm{z}}(Y) \to \mathrm{C}_{\mathrm{z}}(\Sigma_{\mathrm{z}}(f))$ up to homotopy. Since $\mathrm{C}_{\mathrm{z}}(\Sigma_{\mathrm{z}}(f)) \cong \Sigma_{\mathrm{z}}(\mathrm{C}_{\mathrm{z}}(f))$ by Lemma 3.3.11, this can be replaced by $\Sigma_{\mathrm{z}}(f_1) : \Sigma_{\mathrm{z}}(Y) \to \Sigma_{\mathrm{z}} \mathrm{C}_{\mathrm{z}}(f)$, the suspension of the morphism $f_1 : Y \to \mathrm{C}_{\mathrm{z}}(f)$. Continuing in this manner, all iterated mapping cones can be replaced by more familiar homotopy equivalent spaces. Hence, up to homotopy, the coexact sequence (3.3.6) can be replaced by the coexact sequence (3.3.5). □

## 3.4 Slice-coslice categories relative to different base-objects

We finish by comparing the slice-coslice categories for two different fixed base-objects.

**3.4.1 Lemma.** *Suppose $\beta : Z \to Z'$ in $\mathbf{C}$. There exists an adjunction*

$$\begin{tikzcd} \mathbf{C}_Z \arrow[r, bend left, "\beta_*"] \arrow[r, phantom, "\perp"] & \mathbf{C}_{Z'} \arrow[l, bend left, "\beta^*"] \end{tikzcd}$$

*given by $\beta_*(\_) = \_ \vee_Z Z'$ and $\beta^*(\_) = \_ \times_{Z'} Z$, i.e. where $\beta_* X$ and $\beta^* X'$ are defined by the diagrams:*

$$\begin{tikzcd} Z \arrow[r, "\beta"] \arrow[d, "i_X"'] \arrow[dr, phantom, "\ulcorner" very near end] & Z' \arrow[d] \\ X \arrow[r, "f_c"] \arrow[d, "p_X"'] \arrow[dr, phantom, "\ulcorner" very near end] & \beta_* X \arrow[d] \\ Z \arrow[r, "\beta"'] & Z' \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} Z \arrow[r, "\beta"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & Z' \arrow[d, "i_{X'}"] \\ \beta^* X' \arrow[r, "f_p"'] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & X' \arrow[d, "p_{X'}"] \\ Z \arrow[r, "\beta"'] & Z'. \end{tikzcd}$$

*Proof.*

**Claim.** $\mathbf{C}_{Z'}(\beta_* X, X') \cong \{g \in \mathbf{C}(X, X') \mid g i_X = \beta i_{X'} \text{ and } p_{X'} g = \beta p_X\}$.

Suppose we have $\gamma : X \vee_Z Z' \longrightarrow X'$ in $\mathbf{C}_{Z'}(\beta_* X, X')$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "\beta"] \arrow[d, "i_X"'] \arrow[dr, phantom, "\ulcorner" very near end] & Z' \arrow[r, "id_{Z'}"] \arrow[d] & Z' \arrow[d, "i_{X'}"] \\ X \arrow[r, "f_c"] \arrow[d, "p_X"'] \arrow[dr, phantom, "\ulcorner" very near end] & X \vee_Z Z' \arrow[r, "\gamma"] \arrow[d] & X' \arrow[d, "p_{X'}"] \\ Z \arrow[r, "\beta"'] & Z' \arrow[r, "id_{Z'}"'] & Z' \end{tikzcd}$$

Then taking $g = \gamma \circ f_c : X \longrightarrow X'$, clearly

$$\begin{tikzcd} Z \arrow[r, "\beta"] \arrow[d, "i_X"'] & Z' \arrow[d, "i_{X'}"] \\ X \arrow[r, "g"'] \arrow[d, "p_X"'] & X' \arrow[d, "p_{X'}"] \\ Z \arrow[r, "\beta"'] & Z' \end{tikzcd}$$

commutes. Thus $\gamma \in \mathbf{C}_{Z'}(\beta_* X, X')$ induces $g \in \mathbf{C}(X, X')$ such that $gi_{X} = \beta i_{X'}$ and $p_{X'}g = \beta p_{X}$.

Conversely, suppose we have $g \in \mathbf{C}(X, X')$ such that $gi_{X} = \beta i_{X'}$ and $p_{X'}g = \beta p_{X}$. Consider the following diagram

$$\begin{tikzcd}
Z \arrow[r, "\beta"] \arrow[d, "i_X"'] & Z' \arrow[r, "id_{Z'}"] \arrow[d] & Z' \arrow[d, "i_{X'}"] \\
X \arrow[r, "f_c"] \arrow[d, "p_X"'] \arrow[ur, phantom, "\ulcorner" near end] & X \vee_Z Z' \arrow[r, dashed, "\exists!\gamma"] \arrow[d] & X' \arrow[d, "p_{X'}"] \\
Z \arrow[r, "\beta"'] \arrow[ur, phantom, "\ulcorner" near end] & Z' \arrow[r, "id_{Z'}"'] & Z'
\end{tikzcd}$$

where the morphism $\gamma : X \vee_Z Z' \to X'$ is obtained from the universal property of the pushout and the existence of the morphism $g : X \to X'$. An easy check shows that the top and bottom right hand squares commute. Hence

$$\gamma \in \mathbf{C}_{Z'}(X \vee_Z Z', X').$$

**Claim.** $\mathbf{C}_Z(X, \beta^* X') \cong \{g \in \mathbf{C}(X, X') \mid gi_X = \beta i_{X'} \text{ and } p_{X'}g = \beta p_X\}$

Suppose we have $\gamma : X \longrightarrow X' \times_{Z'} Z$ in $\mathbf{C}_Z(X, \beta^* X')$ such that the following diagram commutes:

$$\begin{tikzcd}
Z \arrow[r, "id_Z"] \arrow[d, "i_X"'] & Z \arrow[r, "\beta"] \arrow[d] \arrow[dr, phantom, "\lrcorner" near start] & Z' \arrow[d, "i_{X'}"] \\
X \arrow[r, "\gamma"'] \arrow[d, "p_X"'] & X' \times_{Z'} Z \arrow[r, "f_p"'] \arrow[d] \arrow[dr, phantom, "\lrcorner" near start] & X' \arrow[d, "p_{X'}"] \\
Z \arrow[r, "id_Z"'] & Z \arrow[r, "\beta"'] & Z'.
\end{tikzcd}$$

Then taking $g = f_p \circ \gamma : X \longrightarrow X'$, clearly

$$\begin{tikzcd}
Z \arrow[r, "\beta"] \arrow[d, "i_X"'] & Z' \arrow[d, "i_{X'}"] \\
X \arrow[r, "g"'] \arrow[d, "p_X"'] & X' \arrow[d, "p_{X'}"] \\
Z \arrow[r, "\beta"'] & Z'
\end{tikzcd}$$

commutes. Thus $\gamma \in \mathbf{C}_Z(X, \beta^* X')$ induces $g \in \mathbf{C}(X, X')$ such that $gi_{X} = \beta i_{X'}$ and $p_{X'}g = \beta p_{X}$.

Conversely, suppose we have $g \in \mathbf{C}(X, X')$ such that $gi_{X} = \beta i_{X'}$ and $p_{X'}g = \beta p_{X}$. Consider the following diagram

$$\begin{tikzcd}
Z \arrow[r, "id_Z"] \arrow[d, "i_X"'] & Z \arrow[r, "\beta"] \arrow[d, "i_X"'] \arrow[dr, phantom, "\lrcorner"] & Z' \arrow[d, "i_{X'}"] \\
X \arrow[r, dashed, "\exists!\gamma"'] \arrow[d, "p_X"'] & X' \times_{Z'} Z \arrow[r, "f_p"'] \arrow[d, "p_X"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "p_{X'}"] \\
Z \arrow[r, "id_Z"'] & Z \arrow[r, "\beta"'] & Z'
\end{tikzcd}$$

where the morphism $\gamma : X \to X' \times_{Z'} Z$ is obtained from the universal property of the pullback and the existence of the morphism $g : X \to X'$. It is not difficult to check that the top and bottom left hand squares commute. Hence $\gamma \in \mathbf{C}_Z(X, X' \times_{Z'} Z)$.

Therefore $\mathbf{C}_{Z'}(\beta_* X, X') \cong \mathbf{C}_Z(X, \beta^* X')$, i.e. $\beta_*$ and $\beta^*$ are adjoint. $\square$

**3.4.2 Corollary.**

(i) *The functor $\beta_*$ is right exact, and so it preserves colimits. In particular, $\beta_*$ preserves pushouts and*

$$\beta_*(X \vee_Z Y) \cong \beta_* X \vee_{Z'} \beta_* Y.$$

(ii) *The functor $\beta^*$ is left exact, and so it preserves limits. In particular $\beta^*$ preserves products and*

$$\beta^*(X' \times_{Z'} Y') \cong \beta^* X' \times_Z \beta^* Y'.$$

*Proof.* This follows because left and right adjoints are respectively right and left exact (the basic result from category theory that tells us that right adjoints preserve limits and, dually, left adjoints preserve all colimits can be found in, for example, [Lei14, p.159]). $\square$

**3.4.3 Lemma.** *$\beta^* : \mathbf{C}_{Z'} \to \mathbf{C}_Z$ preserves smash products, i.e.*

$$\beta^*(X' \wedge_{Z'} Y') \cong \beta^* X' \wedge_Z \beta^* Y'.$$

*Proof.* Consider the pushout square defining the smash product of $X'$ and $Y'$ in $\mathbf{C}_{Z'}$

$$\begin{tikzcd}
X' \vee_{Z'} Y' \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Z' \arrow[d] \\
X' \times_{Z'} Y' \arrow[r] & X' \wedge_{Z'} Y'.
\end{tikzcd}$$

Recall that $\beta^*(\_) = Z \times_{Z'} (\_)$. By Corollary 3.4.2, (ii), $\beta^*$ preserves pushouts. Thus, applying $\beta^*$ we obtain a pushout

$$\begin{tikzcd}
(Z \times_{Z'} X') \vee_Z (Z \times_{Z'} Y') \arrow[r, "\beta^* p"] \arrow[d] & Z \arrow[d] \\
(Z \times_{Z'} X') \times_Z (Z \times_{Z'} Y') \arrow[r] \arrow[ur, phantom, "\ulcorner"] & (Z \times_{Z'} X') \wedge_Z (Z \times_{Z'} Y'),
\end{tikzcd}$$

since

$$\beta^*(Z') = Z \times_{Z'} Z' \cong Z,$$

$$\beta^*(X' \vee_{Z'} Y') = Z \times_{Z'} (X' \vee_{Z'} Y') \cong (Z \times_{Z'} X') \vee_Z (Z \times_{Z'} Y'),$$

$$\beta^*(X' \times_{Z'} Y') = Z \times_{Z'} (X' \times_{Z'} Y') \cong (Z \times_{Z'} X') \times_Z (Z \times_{Z'} Y'),$$

$$\beta^*(X' \wedge_{Z'} Y') = Z \times_{Z'} (X' \wedge_{Z'} Y') \cong (Z \times_{Z'} X') \wedge_Z (Z \times_{Z'} Y').$$

Therefore, by the definition of $\beta^*$, we have

$$\beta^*(X' \wedge_{Z'} Y') \cong (Z \times_{Z'} X') \wedge_Z (Z \times_{Z'} Y') = \beta^* X' \wedge_Z \beta^* Y'.$$

□

**3.4.4 Corollary.**

*(i) $\beta^*$ preserves the interval object, i.e. $\beta^*(I_{Z'}) \cong I_Z$, and the interval with disjoint basepoint, i.e.*

$$\beta^*(I^+_{Z'}) \cong I^+_Z.$$

*(ii) $\beta^*$ preserves cones on objects: for any $X' \in \mathbf{C}_{Z'}$*

$$\beta^* \mathrm{C}_{Z'}(X') \cong \mathrm{C}_Z(\beta^* X').$$

*(iii) $\beta^*$ preserves mapping cones: for any $f : X' \to Y'$ in $\mathbf{C}_{Z'}$*

$$\beta^* \mathrm{C}_{Z'}(f) \cong \mathrm{C}_Z(\beta^* f).$$

*(iv) $\beta^*$ preserves suspensions on objects: for any $X' \in \mathbf{C}_{Z'}$*

$$\beta^* \Sigma_{Z'}(X') \cong \Sigma_Z(\beta^* X').$$

*(v) $\beta^*$ preserves homotopies.*

*Proof.*

(i) We have $\beta^*(I_{Z'}) = \beta^*(I \times_{Z'} Z') = Z \times_{Z'} (I \times_{Z'} Z') \cong I \times_Z Z = I_Z$. So $\beta^*(I_{Z'}) = I_Z$. Since

$$\beta^*(I^+_{Z'}) = \beta^*(I \times_{Z'} Z' \vee_\emptyset Z') = Z \times_{Z'} (I \times_{Z'} Z' \vee_\emptyset Z'),$$

and

$$Z \times_{Z'} (I \times_{Z'} Z' \vee_\emptyset Z') \cong Z \times_{Z'} ((I \times_{Z'} Z') \vee_\emptyset (Z \times_{Z'} Z')) \cong (I \times_Z Z) \vee_\emptyset Z = I^+_Z,$$

we have $\beta^*(I^+_{Z'}) \cong I^+_Z$.

(ii) We have $\beta^* \mathrm{C}_{Z'}(X') = \beta^*(X' \wedge_{Z'} I_{Z'})$ and $\mathrm{C}_Z(\beta^* X') = \beta^* X' \wedge_Z I_Z$. By Lemma 3.4.3,

$$\beta^*(X' \wedge_{Z'} I_{Z'}) \cong \beta^* X' \wedge_Z \beta^* I_{Z'}.$$

Since $\beta^* I_{Z'} \cong ((I \times Z') \times_{Z'} Z) \cong I \times Z = I_Z$, we obtain $\beta^* \mathrm{C}_{Z'}(X') \cong \mathrm{C}_Z(\beta^* X')$.

(iii) Suppose $f : X' \to Y'$ in $\mathbf{C}_{Z'}$. Recall that $\beta^*(\_) = (\_) \times_{Z'} Z$. So, by **P.2'**, applying $\beta^*$ to the pushout square defining $\mathrm{C}_{Z'}(f)$ preserves the pushout:

$$\begin{tikzcd}
\beta^* X' \arrow[r, "\beta^* f"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \beta^* Y' \arrow[d] \\
\beta^* \mathrm{C}_{Z'}(X') \arrow[r] & \beta^* \mathrm{C}_{Z'}(f).
\end{tikzcd}$$

Note that $\beta^* \mathrm{C}_{Z'}(X') \cong \mathrm{C}_Z(\beta^* X')$ by (ii). Comparing this with the pushout square defining $\mathrm{C}_Z(\beta^* f)$,

$$\begin{tikzcd}
\beta^* X' \arrow[r, "\beta^* f"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \beta^* Y' \arrow[d] \\
\mathrm{C}_Z(\beta^* X') \arrow[r] & \mathrm{C}_Z(\beta^* f),
\end{tikzcd}$$

we see that $\beta^* \mathrm{C}_{Z'}(f) \cong \mathrm{C}_Z(\beta^* f)$, as required.

(iv) Recall that $\Sigma_{Z'}(X') = \mathrm{C}_{Z'}(p_{X'})$ where $p_{X'} : X' \to Z'$. So

$$\beta^* \Sigma_{Z'}(X') = \beta^* \mathrm{C}_{Z'}(p_{X'}),$$

and by (iii)

$$\beta^* \Sigma_{Z'}(X') = \beta^* \mathrm{C}_{Z'}(p_{X'}) \cong \mathrm{C}_Z(\beta^* p_{X'}),$$

where $\beta^* p_{X'} : \beta^* X' \longrightarrow \beta^* Z' = Z$. Also

$$\Sigma_Z(\beta^* X') = \mathrm{C}_Z(p_{\beta^* X'})$$

where $p_{\beta^* X'} : \beta^* X' \to Z$, i.e. $p_{\beta^* X'} = \beta^* p_{X'}$. Therefore

$$\Sigma_Z(\beta^* X') = \mathrm{C}_Z(\beta^* p_{X'}) \cong \beta^* \Sigma_{Z'}(X').$$

(v) Suppose $H : X' \wedge_{Z'} I^+_{Z'} \longrightarrow Y'$ is a homotopy in $\mathbf{C}_{Z'}$ from $f : X' \to Y'$ to $g : X' \to Y'$, i.e. $H$ is a morphism such that

$$\begin{tikzcd} X' \vee_{Z'} X' \arrow[rr, "f \vee_{Z'} g"] \arrow[dr, "c^-_{Z'}"'] & & Y' \\ & X' \wedge_{Z'} I^+_{Z'} \arrow[ur, dashed, "H"'] & \end{tikzcd}$$

commutes. Applying $\beta^*$ to this triangle, we obtain

$$\begin{tikzcd} \beta^*(X' \vee_{Z'} X') \arrow[rr, "\beta^*(f \vee_{Z'} g)"] \arrow[dr, "\beta^* c^-_{Z'}"'] & & \beta^* Y'. \\ & \beta^*(X' \wedge_{Z'} I^+_{Z'}) \arrow[ur, dashed, "\beta^* H"'] & \end{tikzcd} \tag{3.4.1}$$

We have

$$\beta^*(X' \vee_{Z'} X') = Z \times_{Z'} (X' \vee_{Z'} X') \cong (Z \times_{Z'} X') \vee_Z (Z \times_{Z'} X') = \beta^* X' \vee_Z \beta^* X'.$$

By Lemma 3.4.3, $\beta^*(X' \wedge_{Z'} I^+_{Z'}) \cong \beta^* X' \wedge_Z \beta^* I^+_{Z'}$, and by (i), $\beta^* I^+_{Z'} \cong I^+_Z$. So

$$\beta^*(X' \wedge_{Z'} I^+_{Z'}) \cong \beta^* X' \wedge_Z I^+_Z.$$

Therefore triangle (3.4.1) can be rewritten as

$$\begin{tikzcd} \beta^* X' \vee_Z \beta^* X' \arrow[rr, "\beta^* f \vee_Z \beta^* g"] \arrow[dr, "\beta^* c^-_{Z'}"'] & & \beta^* Y'. \\ & \beta^* X' \wedge_Z I^+_Z \arrow[ur, dashed, "\beta^* H"'] & \end{tikzcd}$$

Thus $\beta^* H$ is a homotopy in $\mathbf{C}_Z$ from $\beta^* f$ to $\beta^* g$.

□

**3.4.5 Lemma.** $\beta_* : \mathbf{C}_Z \to \mathbf{C}_{Z'}$ *preserves smash product with* $(Z \times A)$ *for any* $A \in \mathbf{C}$, *i.e.*

$$\beta_*(X \wedge_Z (Z \times A)) \cong (\beta_* X) \wedge_{Z'} (Z' \times A).$$

*Proof.* Consider

$$
\begin{tikzcd}
X \vee_Z (Z \times A) \arrow[r, "p"] \arrow[d] & Z \arrow[r, "\beta"] \arrow[d] \arrow[dl, phantom, "\ulcorner" very near end] & Z' \arrow[d] \arrow[dl, phantom, "\ulcorner" very near end] \\
X \times_Z (Z \times A) \arrow[r] & X \wedge_Z (Z \times A) \arrow[r] & \beta_*(X \wedge_Z (Z \times A))
\end{tikzcd}
$$

where the left hand square is the pushout defining $X \wedge_Z (Z \times A)$ and the right hand square is the pushout defining $\beta_*(X \wedge_Z (Z \times A))$. By the pasting law for pushouts, the outer rectangle is a pushout, i.e.

$$
\begin{tikzcd}
X \vee_Z (Z \times A) \arrow[r, "\beta p"] \arrow[d] & Z' \arrow[d] \arrow[dl, phantom, "\ulcorner" very near end] \\
X \times_Z (Z \times A) \arrow[r] & \beta_*(X \wedge_Z (Z \times A)).
\end{tikzcd}
\tag{3.4.2}
$$

For comparison, consider the pushout square defining $(\beta_* X) \wedge_{Z'} (Z' \times A)$ as below:

$$
\begin{tikzcd}
\beta_* X \vee_{Z'} (Z' \times A) \arrow[r, "p"] \arrow[d] & Z' \arrow[d] \arrow[dl, phantom, "\ulcorner" very near end] \\
\beta_* X \times_{Z'} (Z' \times A) \arrow[r] & \beta_* X \wedge_{Z'} (Z' \times A).
\end{tikzcd}
$$

Recall that $\beta_*(\_) = Z' \vee_Z (\_)$. Since

$$
(\beta_* X) \vee_{Z'} (Z' \times A) = (Z' \vee_Z X) \vee_{Z'} (Z' \times A) \cong X \vee_Z (Z' \times A)
$$

and

$$
(\beta_* X) \times_{Z'} (Z' \times A) = (Z' \vee_Z X) \times A,
$$

we can rewrite this pushout square as

$$
\begin{tikzcd}
X \vee_Z (Z' \times A) \arrow[r, "p"] \arrow[d] & Z' \arrow[d] \arrow[dl, phantom, "\ulcorner" very near end] \\
(Z' \vee_Z X) \times A \arrow[r] & \beta_* X \wedge_{Z'} (Z' \times A).
\end{tikzcd}
\tag{3.4.3}
$$

Now consider

$$
\begin{tikzcd}
 & & X \vee_Z (Z' \times A) \arrow[drr] \arrow[dd] & & \\
X \vee_Z (Z \times A) \arrow[urr] \arrow[rrrr] \arrow[ddd] \arrow[dr, phantom, "1"] & & & \arrow[dr, phantom, "2"] & Z' \arrow[ddd] \\
 & & (Z' \vee_Z X) \times A \arrow[drr, dashed, "\exists!"] & & \\
X \times_Z (Z \times A) \arrow[urr] \arrow[rrrr] & & & & \beta_*(X \wedge_Z (Z \times A)),
\end{tikzcd}
$$

where $X \times_Z (Z \times A) \cong X \times A$. The front face is the pushout (3.4.2). We now show that square 1 is a pushout. Consider

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "i_X"'] \arrow[dr, phantom, "\ulcorner" very near end] & Z \times A \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & Z' \times A \arrow[d] \\
X \arrow[r] & X \vee_Z (Z \times A) \arrow[r, "\gamma"] \arrow[d] \arrow[dr, phantom, "1"] & X \vee_Z (Z' \times A) \arrow[d] \\
 & X \times A \arrow[r] & (X \vee_Z Z') \times A.
\end{tikzcd}$$

The top left hand square is the pushout defining $X \vee_Z (Z \times A)$ and the top horizontal rectangle is the pushout defining $X \vee_Z (Z' \times A)$. So, by the pasting law, the top right hand square is a pushout. Also, the right hand vertical rectangle is a pushout since taking the product with $A$ of the pushout defining $(X \vee_Z Z')$ preserves the pushout. Applying the pasting law again, we see that square 1 is also a pushout.

Similarly, 2 is a pushout square by the pasting law since the front square (3.4.2) and square 1 are pushouts. Then comparing pushout 2 and the pushout (3.4.3) defining $\beta_* X \wedge_{Z'} (Z' \times A)$, we deduce that

$$\beta_*(X \wedge_Z (Z \times A)) \cong \beta_* X \wedge_{Z'} (Z' \times A).$$

$\square$

**3.4.6 Corollary.**

*(i) $\beta_*$ preserves the interval object, i.e. $\beta_*(I_Z) \cong I_{Z'}$, and the interval with disjoint basepoint, i.e.*

$$\beta_*(I_Z^+) \cong I_{Z'}^+.$$

*(ii) $\beta_*$ preserves cones on objects: for any $X \in \mathbf{C}_Z$*

$$\beta_* \mathrm{C}_Z(X) \cong \mathrm{C}_{Z'}(\beta_* X).$$

*(iii) $\beta_*$ preserves suspensions on objects: for any $X \in \mathbf{C}_Z$*

$$\beta_* \Sigma_Z(X) \cong \Sigma_{Z'}(\beta_* X).$$

*(iv) $\beta_*$ preserves mapping cones: for any $f : X \to Y$ in $\mathbf{C}_Z$*

$$\beta_* \mathrm{C}_Z(f) \cong \mathrm{C}_{Z'}(\beta_* f).$$

*(v) $\beta_*$ preserves homotopies.*

*Proof.*

(i) Since $\beta_*(I_Z) = (I \times_Z Z) \vee_Z Z' = I \times_{Z \vee_Z Z'} (Z \vee_Z Z') = I \times_{Z'} Z'$, we have $\beta_*(I_Z) = I_{Z'}$. The object $I_Z^+$ in $\mathbf{C}_Z$ is defined via

$$\begin{tikzcd} \emptyset \arrow[r, "\iota_Z"] \arrow[d, "\iota_{I_Z}"'] & Z \arrow[d] \\ I \times Z \arrow[r] & I_Z^+ . \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Since $\beta_*(\_) = (\_) \vee_Z Z'$ preserves pushouts by Corollary 3.4.2, (i), the following square is also a pushout and defines $\beta_*(I_Z^+)$

$$\begin{tikzcd} \emptyset \arrow[r, "\beta_*\iota_Z"] \arrow[d, "\beta_*\iota_{I_Z}"'] & Z \vee_Z Z' \arrow[d] \\ I_Z \vee_Z Z' \arrow[r] & I_Z^+ \vee_Z Z'. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We have $Z \vee_Z Z' \cong Z'$ and $I_Z \vee_Z Z' = (I \times Z) \vee_Z Z' \cong I \times Z'$. Thus

$$\beta_* I_Z^+ \cong (I \times Z') \vee_\emptyset Z' \cong I_{Z'}^+ .$$

(ii) We have

$$\beta_* \mathrm{C}_Z(X) = \beta_*(X \wedge_Z I_Z) = \beta_*(X \wedge_Z (I \times Z)),$$

and by Lemma 3.4.5

$$\beta_*(X \wedge_Z (I \times Z)) \cong \beta_* X \wedge_{Z'} (I \times Z').$$

Since

$$\mathrm{C}_{Z'}(\beta_* X) = \beta_* X \wedge_{Z'} I_{Z'} = \beta_* X \wedge_{Z'} (I \times Z'),$$

we see that $\beta_* \mathrm{C}_Z(X) \cong \mathrm{C}_{Z'}(\beta_* X)$.

(iii) Suppose $f : X \to Y$ in $\mathbf{C}_Z$. Applying $\beta_*(\_) = (\_) \vee_Z Z'$ to the pushout square defining $\mathrm{C}_Z(f)$ preserves the pushout (by Lemma 3.4.2, (i)), i.e.

$$\begin{tikzcd} \beta_* X \arrow[r, "\beta_* f"] \arrow[d] & \beta_* Y \arrow[d] \\ \beta_* \mathrm{C}_Z(X) \arrow[r] & \beta_* \mathrm{C}_Z(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where, by (ii), $\beta_* \mathrm{C}_Z(X) \cong \mathrm{C}_{Z'}(\beta_* X)$. Consider $\mathrm{C}_{Z'}(\beta_* f)$ which is defined via the pushout

$$\begin{tikzcd}
\beta_* X \arrow[r, "\beta^* f"] \arrow[d] & \beta_* Y \arrow[d] \\
\mathrm{C}_{Z'}(\beta_* X) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{C}_{Z'}(\beta_* f).
\end{tikzcd}$$

Therefore $\beta_* \,\mathrm{C}_{Z}(f) \cong \mathrm{C}_{Z'}(\beta_* f)$.

(iv) Recall that $\Sigma_Z(X) = \mathrm{C}_Z(p_X)$ where $p_X : X \to Z$. So $\beta_* \Sigma_Z(X) = \beta_* \,\mathrm{C}_Z(p_X)$, and by (iv)

$$\beta_* \Sigma_Z(X) = \beta_* \,\mathrm{C}_Z(p_X) \cong \mathrm{C}_{Z'}(\beta_* p_X),$$

where $\beta_* p_X : \beta_* X \longrightarrow \beta_* Z = Z'$. Also

$$\Sigma_{Z'}(\beta_* X) = \mathrm{C}_{Z'}(p_{\beta_* X})$$

where $p_{\beta_* X} : \beta_* X \to Z'$, i.e. $p_{\beta_* X} = \beta_* p_X$. Therefore

$$\Sigma_{Z'}(\beta_* X) = \mathrm{C}_{Z'}(\beta_* p_X) \cong \beta_* \Sigma_Z(X).$$

(v) Suppose $H : X \wedge_Z I^+_Z \longrightarrow Y$ is a homotopy in $\mathbf{C}_Z$ from $f : X \to Y$ to $g : X \to Y$, i.e. $H$ is a morphism such that

$$\begin{tikzcd}
X \vee_Z X \arrow[rr, "f \vee_Z g"] \arrow[dr, "\bar{c_Z}"'] & & Y \\
& X \wedge_Z I^+_Z \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

commutes. Applying $\beta_*$ to this triangle, we obtain

$$\begin{tikzcd}
\beta_*(X \vee_Z X) \arrow[rr, "\beta_*(f \vee_Z g)"] \arrow[dr, "\beta_* \bar{c_Z}"'] & & \beta_* Y. \\
& \beta_*(X \wedge_Z I^+_Z) \arrow[ur, dashed, "\beta_* H"'] &
\end{tikzcd} \tag{3.4.4}$$

By Lemma 3.4.2, (i), we have $\beta_*(X \vee_Z X) = \beta_* X \vee_{Z'} \beta_* X$. And, by the same lemma, we have

$$\beta_*(X \wedge_Z I^+_Z) \cong \beta_*(X \wedge_Z (I \times Z) \vee_\emptyset Z) \cong \beta_*(X \wedge_Z (I \times Z)) \vee_\emptyset \beta_* Z.$$

Applying Lemma 3.4.5 we have

$$\beta_*(X \wedge_Z I^+_Z) \cong \beta_*(X \wedge_Z (I \times Z)) \vee_\emptyset Z' \cong \beta_* X \wedge_{Z'} (I \times Z') \vee_\emptyset Z',$$

where $(I \times Z') \vee_\emptyset Z' = I^+_{Z'}$. Thus $\beta_*(X \wedge_Z I^+_{Z'}) \cong \beta_* X \wedge_{Z'} I^+_{Z'}$. Therefore triangle (3.4.4) can be rewritten as

$$\begin{tikzcd}
\beta_* X \vee_{Z'} \beta_* X \arrow[rr, "\beta_* f \vee_{Z'} \beta_* g"] \arrow[dr, "\beta_* \bar{c_Z}"'] & & \beta_* Y. \\
& \beta_* X \wedge_{Z'} I^+_{Z'} \arrow[ur, dashed, "\beta_* H"'] &
\end{tikzcd}$$

Thus $\beta_* H$ is a homotopy in $\mathbf{C}_{Z'}$ from $\beta_* f$ to $\beta_* g$.

$\square$

## 3.5 The Spanier–Whitehead category of the slice-coslice category

Fix $Z \in \mathbf{C}$, and consider the slice-coslice category $\mathbf{C}_Z$.

**3.5.1 Definition. The Spanier–Whitehead category of $\mathbf{C}_Z$**, denoted $\mathbf{SW}(\mathbf{C}_Z)$, has as objects pairs $(X, m)$, where $X \in \mathbf{C}_Z$ and $m \in \mathbb{Z}$, and morphisms

$$\mathrm{Hom}_{\mathbf{SW}(\mathbf{C}_Z)}\big((X,m),(Y,n)\big) := \mathrm{colim}_{k\to\infty}[{\Sigma_{\mathrm{Z}}}^{k+m}(X), {\Sigma_{\mathrm{Z}}}^{k+n}(Y)]$$

(where $[P, Q]$ denotes the set of based homotopy classes of morphisms $P \to Q$ in $\mathbf{C}_Z$).

Recall that $(A, m) \cong (\Sigma_{\mathrm{Z}}(A), m-1)$. To simplify notation, when possible we will denote an object $(X, 0)$ of $\mathbf{SW}(\mathbf{C}_Z)$ just as $X$. The induced suspension $\Sigma_{\mathrm{Z}} : \mathbf{SW}(\mathbf{C}_Z) \to \mathbf{SW}(\mathbf{C}_Z)$ is given by $\Sigma_{\mathrm{Z}}(X, m) = (\Sigma_{\mathrm{Z}}(X), m)$.

**3.5.2 Definition.** The **formal suspension** on $\mathbf{SW}(\mathbf{C}_Z)$ is given by the shift functor

$$T(X, m) = (X, m+1).$$

Since $(X, m+1) \cong (\Sigma_{\mathrm{Z}}(X), m)$, there is a natural isomorphism

$$T(X, m) \cong \Sigma_{\mathrm{Z}}(X, m),$$

i.e. suspension is isomorphic to the shift. Hence suspension is invertible in $\mathbf{SW}(\mathbf{C}_Z)$ and one can consider *de-suspensions* of objects in $\mathbf{SW}(\mathbf{C}_Z)$.

**3.5.3 Remark.** There is an evident functor

$$F : \mathbf{C}_Z \to \mathbf{SW}(\mathbf{C}_Z)$$

given by

$$X \mapsto (X, 0)$$

and

$$(X \to Y) \mapsto \operatorname{colim}_{k\to\infty}[\Sigma_Z^k(X), \Sigma_Z^k(Y)].$$

By definition, after sufficiently many suspensions, any morphism in $\mathbf{SW}(\mathbf{C}_Z)$ is isomorphic to one in the image of $F$.

**3.5.4 Lemma.** *Any morphism in $\mathbf{SW}(\mathbf{C}_Z)$ can be represented by a cofibration in $\mathbf{C}_Z$. More precisely, for any morphism*

$$(A, m) \xrightarrow{\phi} (B, n)$$

*in $\mathbf{SW}(\mathbf{C}_Z)$, one can find a cofibration $\sigma$ in $\mathbf{C}_Z$ and some $k \in \mathbb{N}$ such that, up to composing with an isomorphism, $\phi$ is given by*

$$\Sigma_{\mathrm{z}}^{\text{-k}}(F(\sigma))$$

*where $F(\sigma)$ denotes the image of $\sigma$ under the functor $F : \mathbf{C}_Z \to \mathbf{SW}(\mathbf{C}_Z)$ given by $X \mapsto (X, 0)$.*

*Proof.* By definition, $\phi$ is represented by some morphism

$$f : \Sigma_{\mathrm{z}}^{\mathrm{k+m}}(A) \to \Sigma_{\mathrm{z}}^{\mathrm{k+n}}(B)$$

in $\mathbf{C}_Z$ for sufficiently large $k \in \mathbb{N}$, in particular sufficiently large that we have $k + m, k + n \geq 0$. Rephrasing, using the fact that

$$(\Sigma_{\mathrm{z}}^{\mathrm{k+m}}(A), 0) \cong \Sigma_{\mathrm{z}}^{\mathrm{k}}(\Sigma_{\mathrm{z}}^{\mathrm{m}}(A), 0) \cong \Sigma_{\mathrm{z}}^{\mathrm{k}}((A), m),$$

we have

$$\phi = \Sigma_{\mathrm{z}}^{\text{-k}}(F(f)).$$

However, there is a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma_{\mathrm{z}}^{\mathrm{k+m}}(A) \arrow[r, "f"] \arrow[dr, "\sigma"'] & \Sigma_{\mathrm{z}}^{\mathrm{k+n}}(B) \arrow[d] \\
& \mathrm{Cyl}_{\mathrm{z}}(f)
\end{tikzcd}$$

in $\mathbf{C}_Z$, in which $\sigma$ is a cofibration by Lemma 3.2.25. From Lemma 3.2.32 we know that $B \to \mathrm{Cyl}_{\mathrm{z}}(f)$ is a homotopy equivalence in $\mathbf{C}_Z$. Thus in $\mathbf{SW}(\mathbf{C}_Z)$ the morphism $\Sigma_{\mathrm{z}}^{\mathrm{k+n}}(B) \to \mathrm{Cyl}_{\mathrm{z}}(f)$ is an isomorphism. The result follows. □

**3.5.5 Definition.** The **exact triangles** in $\mathbf{SW}(\mathbf{C}_Z)$ are the mapping sequences $X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C}_{\mathrm{z}}(f) \xrightarrow{f_2} \Sigma_{\mathrm{z}}(X)$ defined by shortened coexact Puppe sequences as in (3.3.5), and any sequence isomorphic to one of these.

Thus, by definition, an exact triangle in $\mathbf{SW}(\mathbf{C}_Z)$ is any diagram which is isomorphic to an iterated (de)-suspension of the image of a sequence

$$A \xrightarrow{f} B \to \mathrm{C}_{\mathrm{z}}(f) \to \Sigma_{\mathrm{z}}(A)$$

in $\mathbf{C}_Z$ under the functor $F$. As in Lemma 3.5.4, we can replace $f$ by a cofibration, only altering the image diagram in $\mathbf{SW}(\mathbf{C}_Z)$ by an isomorphism. Hence we may always assume that $f$ is a cofibration if we wish. Moreover, in this case, the sequence above is isomorphic in $\mathbf{SW}(\mathbf{C}_Z)$ to the diagram

$$A \xrightarrow{f} B \to B/A \to \Sigma_{\mathrm{z}}(A)$$

using Lemma 3.3.7 which tells us that $\mathrm{C}_{\mathrm{z}}(f) \cong B/A$ in $\mathbf{SW}(\mathbf{C}_Z)$.

**3.5.6 Definition.** The **collection of distinguished triangles,** $\Delta$ consists of exact triangles which are the diagrams of the form $\mathbf{A} \to \mathbf{B} \to \mathbf{C} \to \Sigma_{\mathrm{z}}(\mathbf{A})$, i.e. mapping sequences (and their supensions and de-suspensions). Hence, any sequence in $\mathbf{SW}(\mathbf{C}_Z)$ of the form $A \to B \to C \to \Sigma_{\mathrm{z}}(A)$ that is isomorphic to a mapping sequence $X \xrightarrow{f} Y \xrightarrow{f_1} \mathrm{C}_{\mathrm{z}}(f) \xrightarrow{f_2} \Sigma_{\mathrm{z}}(X)$ is a distinguished triangle.

**3.5.7 Lemma.** $\mathbf{SW}(\mathbf{C}_Z)$ *is an additive category.*

*Proof.* This follows from the fact that the abelian group structure on $[\Sigma_{\mathrm{z}}^2(X), Y]$ is natural (by Corollary 3.2.36) and the fact that

$$\mathrm{Hom}_{\mathbf{SW}(\mathbf{C}_Z)}\big((X,m),(Y,n)\big) := \mathrm{colim}_{k\to\infty}[\Sigma_{\mathrm{z}}{}^{k+m}(X), \Sigma_{\mathrm{z}}{}^{k+n}(Y)].$$

□

**3.5.8 Theorem.** *Suppose* $\mathbf{C}$ *is a category satisfying properties* ***P.1-P.5****, and fix* $Z \in \mathbf{C}$*. Then the Spanier–Whitehead category,* $\mathbf{SW}(\mathbf{C}_Z)$*, is a triangulated category.*

*Proof.* We verify that $(\mathbf{SW}(\mathbf{C}_Z), \Sigma_{\mathrm{z}}, \Delta)$ satisfies the axioms of a triangulated category.

**TR.1** $\boldsymbol{\Delta}$ **is replete:** By definition $\Delta$ is a collection of distinguished triangles consisting of mapping sequences and their (de-)suspensions. Any sequence in $\mathbf{SW}(\mathbf{C}_Z)$ of the form $A \to B \to C \to \Sigma_{\mathrm{z}}(A)$ that is isomorphic to a distinguished triangle is also a distinguished triangle. Thus, by definition, $\Delta$ is replete.

**TR.2 For all X in** $\mathrm{Ob}(\mathbf{SW}(\mathbf{C_Z}))$**, the triangle** $\mathbf{Z} \xrightarrow{i_x} \mathbf{X} \xrightarrow{\mathbf{id_X}} \mathbf{X} \to \mathbf{Z}$ **is in** $\boldsymbol{\Delta}$**:** consider the mapping sequence $Z \xrightarrow{i_x} X \xrightarrow{f_1} \mathrm{C}_{\mathrm{z}}(i_x) \xrightarrow{f_2} \mathrm{C}_{\mathrm{z}}(f_1)$. The pushout defining $\mathrm{C}_{\mathrm{z}}(i_x)$ is

$$\begin{tikzcd} Z \arrow[r, "i_X"] \arrow[d] & X \arrow[d] \\ Z \wedge_Z I_Z \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{C}_Z(i_X), \end{tikzcd}$$

where $Z \wedge_Z I_Z \cong Z$. So we have $\mathrm{C}_Z(i_X) \cong Z \vee_Z X \cong X$, and

$$f_1 = id_X : X \to X.$$

The object $\mathrm{C}_Z(id_X)$ is defined via the pushout

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d] & X \arrow[d] \\ X \wedge_Z I_Z \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{C}_Z(id_X), \end{tikzcd}$$

i.e. $\mathrm{C}_Z(id_X) \cong X \vee_X (X \wedge_Z I_Z) \cong X \wedge_Z I_Z$. Since $I_Z \simeq Z$, we have $\mathrm{C}_Z(id_X) \simeq Z$ in $\mathbf{C}_Z$. Thus $I_Z \cong Z$ in $\mathbf{SW}(\mathbf{C}_Z)$ and $\mathrm{C}_Z(id_X) \cong X \wedge_Z Z \cong Z$. So $Z \xrightarrow{i_X} X \xrightarrow{id_X} X \to Z$ is isomorphic to the mapping sequence

$$Z \xrightarrow{i_X} X \xrightarrow{f_1} \mathrm{C}_Z(i_X) \xrightarrow{f_2} \mathrm{C}_Z(f_1).$$

Hence the triangle $Z \xrightarrow{i_X} X \xrightarrow{id_X} X \to Z$ is in $\Delta$.

**TR.3 For all $f$ in $\mathrm{Hom}_{\mathbf{SW}(\mathbf{C}_Z)}(A,B)$, there exists a distinguished triangle of the form $\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \to \mathbf{C} \to \Sigma_Z(\mathbf{A})$:** this follows immediately from the definition of mapping sequences and the definition of the collection of distinguished triangles.

**TR.4 The triangle $\mathbf{A} \xrightarrow{\mathbf{u}} \mathbf{B} \xrightarrow{\mathbf{v}} \mathbf{C} \xrightarrow{\mathbf{w}} \Sigma_Z(\mathbf{A})$ is in $\boldsymbol{\Delta}$ if and only if the triangle $\mathbf{B} \xrightarrow{\mathbf{v}} \mathbf{C} \xrightarrow{\mathbf{w}} \Sigma_Z(\mathbf{A}) \xrightarrow{-\Sigma_Z(\mathbf{u})} \Sigma_Z(\mathbf{B})$ is in $\boldsymbol{\Delta}$:** recall that a triangle in $\mathbf{SW}(\mathbf{C}_Z)$ comes from an underlying sequence in $\mathbf{C}_Z$ up to isomorphism and suspension, in particular, a triangle is a diagram which is isomorphic to an iterated suspension of the image of a shortened Puppe sequence in $\mathbf{C}_Z$. As in the classical case, any shortened Puppe sequence extends to the right by Corollary 3.3.12. Thus the result follows the same reasoning as in the classical case, taking care with the minus sign of $\Sigma_Z(u)$ which arises from the fact that the transposition $\tau$ is used in Lemma 3.3.9 and induces the inverse in the morphism groups of $\mathbf{SW}(\mathbf{C}_Z)$.

**TR.5 Given triangles $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma_Z(A)$ and $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} \Sigma_Z(A')$ in $\Delta$, and morphisms $f : A \to A'$ and $g : B \to B'$ such that $gu = u'f$, then there exists a fill-in morphism $h$ (not necessarily unique) making all the squares in the following diagram commute:**

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, dashed, "h"] & \Sigma_Z(A) \arrow[d, "\Sigma_Z(f)"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \Sigma_Z(A').
\end{tikzcd}$$

Since, as previously discussed, any triangle in $\mathbf{SW}(\mathbf{C}_Z)$ can be thought of, up to isomorphism, as a shortened Puppe sequence in $\mathbf{C}_Z$, this axiom follows immediately from Lemma 3.3.6.

**TR.6 The octahedral axiom: Given two morphisms $f : A \to B$ and $g : B \to C$ in $\mathbf{SW}(\mathbf{C}_Z)$, the distinguished triangles on $f, g$, and on the composition morphism $gf$ can be formed:**

$$A \xrightarrow{f} B \xrightarrow{f_1} \mathrm{C_Z}(f) \xrightarrow{p(f)} \dots ,$$

$$B \xrightarrow{g} C \xrightarrow{g_1} \mathrm{C_Z}(g) \xrightarrow{p(g)} \dots ,$$

$$A \xrightarrow{gf} C \xrightarrow{(gf)_1} \mathrm{C_Z}(gf) \xrightarrow{p(gf)} \dots .$$

**Then there exists a distinguished triangle**

$$\mathrm{C_Z}(f) \xrightarrow{\gamma} \mathrm{C_Z}(gf) \xrightarrow{\beta} \mathrm{C_Z}(g) \xrightarrow{\alpha} \dots ,$$

**such that the triangles with an odd number of solid arrows in the following diagram commute:**

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[rrrr, bend left, "gf"] & & B \arrow[rr, "g"] \arrow[dl, "f_1"] & & C, \arrow[dl, "g_1"] \arrow[ddll, bend left, "(gf)_1"] \\
& \mathrm{C_Z}(f) \arrow[ul, dashed, "f_2"] \arrow[dr, "\exists!\gamma"] & & \mathrm{C_Z}(g) \arrow[ul, dashed, "g_2"] \arrow[ll, dashed, "\alpha"] & \\
& & \mathrm{C_Z}(gf) \arrow[ur, "\exists!\beta"] \arrow[uull, dashed, bend left, "(gf)_2"] & &
\end{tikzcd}$$

**i.e.**

$g_1 = \beta \circ (gf)_1,$

$f_2 = (gf)_2 \circ \gamma,$

$\alpha = \Sigma(f_1) \circ g_2,$

**and such that** $g_2 \circ \beta = \Sigma(f) \circ (gf)_2$ **and** $\gamma \circ f_1 = (gf)_1 \circ g$**.**

Using the fact that, after suitable (de)-suspension any triangle in $\mathbf{SW}(\mathbf{C}_Z)$ is isomorphic to the image of a sequence $A \xrightarrow{f} B \to \mathrm{C}_{\mathrm{Z}}(f) \to \Sigma_{\mathrm{Z}}(A)$ in $\mathbf{C}_Z$ where $f$ is a cofibration, we reduce to working in $\mathbf{C}_Z$ with both $f$ and $g$ cofibrations. Thus, by Lemma 3.2.22, (ii), the composite $gf : A \to C$ is a cofibration. Also we know that $\epsilon : B/A \to C/A$ is a cofibration (by Lemma 3.2.27). Now consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p_A"'] & B \arrow[r, "g"] \arrow[d] & C \arrow[d] \\
Z \arrow[r] \arrow[ur, phantom, "\ulcorner"] & B/A \arrow[r, "\epsilon"] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & C/A \arrow[d] \\
 & Z \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C/B
\end{tikzcd}$$

where $C/B \cong (C/A)/(B/A)$ from Lemma 3.2.27. Then, by Lemma 3.3.7, we have

$$\mathrm{C}_{\mathrm{Z}}(f) \simeq B/A,$$

$$\mathrm{C}_{\mathrm{Z}}(g) \simeq C/B,$$

and

$$\mathrm{C}_{\mathrm{Z}}(gf) \simeq C/A.$$

Thus the triangle $\mathrm{C}_{\mathrm{Z}}(f) \xrightarrow{\gamma} \mathrm{C}_{\mathrm{Z}}(gf) \xrightarrow{\beta} \mathrm{C}_{\mathrm{Z}}(g)$ arises as (the image in $\mathbf{SW}(\mathbf{C}_Z)$ of) the bottom right hand pushout square, i.e.

$$\begin{tikzcd}
\mathrm{C}_{\mathrm{Z}}(f) \arrow[r, "\gamma"] \arrow[d] & \mathrm{C}_{\mathrm{Z}}(gf) \arrow[d, "\beta"] \\
Z \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{C}_{\mathrm{Z}}(g).
\end{tikzcd}$$

□

**3.5.9 Remark.** The octahedral axiom means that cones of composites behave reasonably.

**3.5.10 Lemma.** $\mathbf{SW}(\mathbf{C_Z})$ *is a tensor triangulated category.*

*Proof.* By Theorem 3.5.8 we know that $\mathbf{SW}(\mathbf{C}_Z)$ is triangulated. We need to show that it has the structure of a (symmetric) monoidal category and of a triangulated category in a compatible way. The tensor product in $\mathbf{SW}(\mathbf{C}_Z)$ is given by the functor

$$\_ \wedge_Z \_ : \mathbf{SW}(\mathbf{C}_Z) \times \mathbf{SW}(\mathbf{C}_Z) \to \mathbf{SW}(\mathbf{C}_Z),$$

defined by

$$(X, m) \wedge_Z (Y, n) = (X \wedge_Z Y, m + n).$$

For objects of the form $(X, m)$ with $m \in \mathbb{N}$, this is compatible with the definition of the smash product in $\mathbf{C}_Z$. From Lemma 2.4.2, we know that

$$(X, m) \cong ({\Sigma_{\mathrm{z}}}^m(X), 0).$$

Writing ${\Sigma_{\mathrm{z}}}^m(X)$ rather than $({\Sigma_{\mathrm{z}}}^m(X), 0)$, we have ${\Sigma_{\mathrm{z}}}^m(X) \cong X \wedge_Z S^m_Z$ and ${\Sigma_{\mathrm{z}}}^n(Y) \cong Y \wedge_Z S^n_Z$ in $\mathbf{C}_Z$ by Lemma 3.2.14. So

$$\begin{aligned}
{\Sigma_{\mathrm{z}}}^m(X) \wedge_Z {\Sigma_{\mathrm{z}}}^n(Y) &\cong (X \wedge_Z S^m_Z) \wedge_Z (Y \wedge_Z S^n_Z)\\
&\cong X \wedge_Z (S^m_Z \wedge_Z S^n_Z) \wedge_Z Y\\
&\cong X \wedge_Z S^{m+n}_Z \wedge_Z Y\\
&\cong (X \wedge_Z Y) \wedge_Z S^{m+n}_Z\\
&\cong {\Sigma_{\mathrm{z}}}^{m+n}(X \wedge_Z Y)
\end{aligned}$$

(using the fact that $S^n_Z := \Sigma^{\mathrm{n}}_{\mathrm{z}}(S^0_Z)$ in $\mathbf{C}_Z$, as in Example 3.2.13, and applying Lemma 3.2.6, (iv) and (i)). Hence

$$\begin{aligned}
(X, m) \wedge_Z (Y, n) &\cong ({\Sigma_{\mathrm{z}}}^m(X), 0) \wedge_Z ({\Sigma_{\mathrm{z}}}^n(Y), 0)\\
&\cong ({\Sigma_{\mathrm{z}}}^{m+n}(X \wedge_Z Y), 0)\\
&\cong (X \wedge_Z Y, m + n).
\end{aligned}$$

So the smash product can be extended to $\mathbf{SW}(\mathbf{C}_Z)$ compatibly with the functor $F$ in Remark 3.5.3. By Lemma 3.2.18, we know that smash product preserves homotopies, mapping cones, and suspensions, and from Lemma 3.2.6, (v), we know that smash product preserves pushouts in $\mathbf{C}_Z$. Hence the smash product functor which gives the category $\mathbf{C}_Z$ a monoidal structure by Corollary 3.2.8, descends to a triangulated functor on $\mathbf{SW}(\mathbf{C}_Z)$ and also gives $\mathbf{SW}(\mathbf{C}_Z)$ a monoidal structure. □

**3.5.11 Corollary.** *$K(\mathbf{SW}(\mathbf{C}_Z))$ is a ring.*

*Proof.* This is a standard consequence of the fact that $\mathbf{SW}(\mathbf{C}_Z)$ is tensor-triangulated (Lemma 3.5.10 above). The sum and the product are induced from $\vee_Z$ and $\wedge_Z$. □

## 3.6 Functoriality

We now prove that there exists a base change functoriality for the relative Spanier–Whitehead categories.

**3.6.1 Theorem.** *Suppose $\beta : Z \to Z'$ in $\mathbf{C}$. The adjunction*

$$\begin{tikzcd} \mathbf{C}_Z \arrow[r, bend left, "\beta_*"] \arrow[r, phantom, "\perp"] & \mathbf{C}_{Z'} \arrow[l, bend left, "\beta^*"] \end{tikzcd}$$

*given by $\beta_*(\_) = \_ \vee_Z Z'$ and $\beta^*(\_) = \_ \times_{Z'} Z$, as in Lemma 3.4.1, descends to an adjunction*

$$\begin{tikzcd} \mathbf{SW}(\mathbf{C}_Z) \arrow[r, bend left, "\beta_*"] \arrow[r, phantom, "\perp"] & \mathbf{SW}(\mathbf{C}_{Z'}) \arrow[l, bend left, "\beta^*"] \end{tikzcd}$$

*where $\beta_*$ and $\beta^*$ are triangulated functors, and $\beta^*$ is monoidal.*

*Proof.* The functors $\beta_* : \mathbf{SW}(\mathbf{C}_Z) \to \mathbf{SW}(\mathbf{C}_{Z'})$ and $\beta^* : \mathbf{SW}(\mathbf{C}_{Z'}) \to \mathbf{SW}(\mathbf{C}_Z)$ are triangulated since they preserve cones on objects, mapping cones, suspensions and homotopies (by Lemma 3.4.6 and Lemma 3.4.4, respectively.) In addition, $\beta^*$ is monoidal since, by Lemma 3.4.5, it preserves the smash product. Now, since $\beta_*$ and $\beta^*$ preserve suspensions and homotopies, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{SW}(\mathbf{C}_{Z'})}\big(\beta_*(X,m),(X',m')\big) &:= \mathrm{colim}_{k\to\infty}[{\Sigma_{Z'}}^{k+m}(\beta_* X), {\Sigma_{Z'}}^{k+m'}(X')] \\ &\cong \mathrm{colim}_{k\to\infty}[\beta_* {\Sigma_Z}^{k+m}(X), {\Sigma_{Z'}}^{k+m'}(X')], \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_{\mathbf{SW}(\mathbf{C}_{Z})}\big((X,m),\beta^*(X',m')\big) &:= \mathrm{colim}_{k\to\infty}[{\Sigma_{Z}}^{k+m}(X), {\Sigma_{Z}}^{k+m'}\beta^*(X')] \\ &\cong \mathrm{colim}_{k\to\infty}[{\Sigma_Z}^{k+m}(X), \beta^* {\Sigma_{Z'}}^{k+m'}(X')]. \end{aligned}$$

By Lemma 3.4.1 we know that

$$\mathrm{colim}_{k\to\infty}[\beta_* {\Sigma_Z}^{k+m}(X), {\Sigma_{Z'}}^{k+m'}(X')] \cong \mathrm{colim}_{k\to\infty}[{\Sigma_Z}^{k+m}(X), \beta^* {\Sigma_{Z'}}^{k+m'}(X')].$$

Therefore

$$\mathrm{Hom}_{\mathbf{SW}(\mathbf{C}_{Z'})}\big(\beta_*(X,m),(X',m')\big) \cong \mathrm{Hom}_{\mathbf{SW}(\mathbf{C}_{Z})}\big((X,m),\beta^*(X',m')\big),$$

i.e. $\beta_*$ is left adjoint to $\beta^*$, as required. □

**3.6.2 Remark.** [Nee01, p.181] states that, given an adjunction between triangulated categories $\mathbf{A}$ and $\mathbf{B}$,

$$\begin{tikzcd} \mathbf{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{B}, \arrow[l, bend left, "G"] \end{tikzcd}$$

then $F$ is triangulated if and only if $G$ is triangulated. Note this is a property which does not hold for abelian categories.

We note that there exists also a change of ambient category functoriality. Specifically, let **C** and **C'** be two categories satisfying properties **P.1-P.5** and suppose a functor $J : \mathbf{C} \to \mathbf{C'}$ preserves the interval object, i.e. $J$ takes

$$\begin{tikzcd} S^0 \arrow[rr, "id\vee_* id"] \arrow[dr, dashed, "c_*"'] & & * \\ & I \arrow[ur, "\pi_I"'] & \end{tikzcd}$$

where $c_*$ is a cofibration, to

$$\begin{tikzcd} J(S^0) \arrow[rr, "id\vee_* id"] \arrow[dr, dashed, "c'_*=J(c_*)"'] & & J(*) = * \\ & J(I) \arrow[ur, "\pi_I"'] & \end{tikzcd}$$

where $c'_*$ is also a cofibration, and suppose $J$ preserves finite limits (i.e. binary products and equalisers) and colimits (i.e. binary coproducts and coequalisers). Then it is clear that $J$ induces a triangulated functor

$$J : \mathbf{SW}(\mathbf{C}_Z) \to \mathbf{SW}(\mathbf{C'}_{J(Z)})$$

for each $Z \in \mathbf{C}$.

# Chapter 4

# The Spanier–Whitehead categories

Our main objective is to provide a homotopical categorification of the Euler calculus. We hope this categorification will be richer than the already known homological categorification of the Euler calculus. It will be achieved by constructing a relative definable version of the Spanier–Whitehead category and showing that its Grothendieck group is exactly the ring of constructible functions over the fixed base-object.

The axiomatic approach to the construction of Spanier–Whitehead categories given some ambient category $\mathbf{C}$ satisfying **P.1 -P.5**, described in Chapter 3, allows us to produce a family of tensor-triangulated Spanier–Whitehead categories $\mathbf{SW}(\mathbf{C}_Z)$ indexed by objects $Z \in \mathbf{C}$.

Fixing an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$, we want to be able to apply the axiomatic approach using the category **Def** to construct the relative definable Spanier–Whitehead categories $\mathbf{SW}(\mathbf{Def}_Z)$. However, it is not clear whether the category **Def** has all finite colimits, and as a result it is not clear whether or not they satisfy the ambient category conditions **P.1** and **P.2**. Thus the formal framework of Chapter 3 does not quite apply for the category **Def**.

Since we do know that pushouts along closed inclusions exist in **Def** by Lemma 1.3.3, and that products commute with pushouts, we can carefully verify that each pushout used in the axiomatic construction of the Spanier–Whitehead category in Chapter 3 is a pushout along a closed inclusion, and hence exists in **Def**. Thus we will show that, by formulating a set of weaker assumptions on the ambient category, we can still use this axiomatic framework to produce tensor-triangulated categories, $\mathbf{SW}(\mathbf{Def}_Z)$, for each $Z \in \mathbf{Def}$. In addition, the set of weaker assumptions will also hold for the category $\mathbf{CW}_*$ of finite pointed CW-complexes. So we can also axiomatically construct the classical Spanier–Whitehead category $\mathbf{SW}(\mathbf{CW}_*)$.

The proof that the Spanier–Whitehead categories are tensor-triangulated follows directly from this axiomatic approach. So, once it has been formally established that **Def** is a satisfactory choice of ambient category, we can immediately define $\mathbf{SW}(\mathbf{Def}_Z)$ for any $Z \in \mathbf{Def}$ as a tensor-triangulated category.

Since the Grothendieck group of a tensor-triangulated category has a ring structure, it follows that the Grothendieck groups $K(\mathbf{SW}(\mathbf{Def}_Z))$ of the relative definable Spanier–Whitehead categories are rings (for each $Z \in \mathbf{Def}$). At the crux of this thesis is the proof that the Grothendieck group of the relative definable Spanier–Whitehead category is exactly the ring of constructible functions over $Z$, denoted $CF(Z)$. This is precisely the homotopical categorification of the constructible functions which enables the operations of the Euler calculus to be lifted to functors between relative definable Spanier–Whitehead categories.

The structure of this chapter is as follows. In Section 4.1 we formulate the set of weaker assumptions on the ambient category of the axiomatically constructed Spanier–Whitehead category which will hold in particular for **Def**. Section 4.2 consists of a verification that $\mathbf{CW}_*$ satisfies these weaker conditions and an alternative method for proving that $\mathbf{SW}(\mathbf{CW}_*)$ is a tensor triangulated category. Then in Section 4.3 we discuss the existence of finite colimits in **Def**, but then show that **Def** does satisfy the weaker ambient category assumptions. In section 4.4 we consider the absolute case where the base-object is fixed to be the point $*$. We give the definition of the definable Spanier–Whitehead category $\mathbf{SW}(\mathbf{Def}_*)$ and the verification that it is tensor-triangulated in this Section. Then, in Section 4.5, we compute the Grothendieck group of $\mathbf{SW}(\mathbf{Def}_*)$ and show that $K(\mathbf{SW}(\mathbf{Def}_*)) \cong \mathbb{Z}$. Section 4.6 introduces the relative definable Spanier–Whitehead category $\mathbf{SW}(\mathbf{Def}_Z)$ where $Z$ is any object in **Def** and provides the proof of its tensor-triangulated structure. The proof of the key result that $K(\mathbf{SW}(\mathbf{Def}_Z)) \cong CF(Z)$ is found in Section 4.7. We then discuss lifting the Euler calculus to $\mathbf{SW}(\mathbf{Def}_Z)$ in Section 4.8.

## 4.1 A weaker set of assumptions on the ambient category

We would now like to apply the approach described in Chapter 3 to construct various Spanier–Whitehead categories. As will be discussed in Section 4.3, it is doubtful whether categories such as **Def** has all finite colimits, and as a result whether or not they satisfy conditions **P.1** and **P.2**.

We have shown that the given conditions, **P.1**, **P.2**, **P.3**, **P.4**, and **P.5**, on the ambient category **C** suffice to construct $\mathbf{SW}(\mathbf{C}_Z)$ for any $Z \in \mathbf{C}$. However, they are stronger than necessary. In particular, we need not assume **C** has all finite colimits; only certain pushouts are required for the Spanier–Whitehead construction. This section contains the proposition formalising the statement of this weaker set of assumptions on the ambient category.

**4.1.1 Proposition.** *Suppose* $\mathbf{C}$ *is a category satisfying* ***P.3***, ***P.4***, ***P.5*** *and such that the following pushouts exist in* $\mathbf{C}$ *and commute with products:*

***A.1*** *the pushout defining the mapping cylinder on a morphism* $f : X \to Y$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id\times\iota_0"'] \arrow[dr, phantom, "\ulcorner" very near end] & Y \arrow[d] \\ X \times I \arrow[r] & \mathrm{Cyl}(f), \end{tikzcd}$$

*in particular the cone on an object* $X \in \mathbf{C}$,

$$C(X) := \mathrm{Cyl}(X \xrightarrow{\pi_X} *);$$

***A.2*** *the pushout defining the mapping cone on a morphism* $f : X \to Y$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id\times\iota_1"'] \arrow[dr, phantom, "\ulcorner" very near end] & Y \arrow[d] \\ C(X) \arrow[r] & C(f), \end{tikzcd}$$

*in particular the suspension of an object* $X \in \mathbf{C}$,

$$S(X) := C(X \xrightarrow{\pi_X} *);$$

***A.3*** *the pushout defining the double interval* $I'$,

$$\begin{tikzcd} * \arrow[r, "\iota_0"] \arrow[d, "\iota_1"'] \arrow[dr, phantom, "\ulcorner" very near end] & I \arrow[d] \\ I \arrow[r] & I'; \end{tikzcd}$$

*and suppose* $Z$ *is an object in* $\mathbf{C}$ *such that the following pushouts exist in* $\mathbf{C}_Z$ *and commute with products:*

***B.1*** *the pushout defining the smash product of* $X$ *and* $Y$,

$$\begin{tikzcd} X \vee_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near end] & X \times_Z Y \arrow[d] \\ Z \arrow[r] & X \wedge_Z Y; \end{tikzcd}$$

***B.2*** *the pushout defining the mapping cylinder on a morphism* $f : X \to Y$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id\wedge_Z v_1"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ \mathrm{Cyl}_{\mathrm{z}}(X) \arrow[r] & \mathrm{Cyl}_{\mathrm{z}}(f), \end{tikzcd}$$

*where* $\mathrm{Cyl}_{\mathrm{z}}(X) := X \wedge_Z I_Z^+$*;*

***B.3*** *the pushout defining the mapping cone on a morphism* $f : X \to Y$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id\wedge_Z c_1"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ \mathrm{C}_{\mathrm{z}}(X) \arrow[r] & \mathrm{C}_{\mathrm{z}}(f), \end{tikzcd}$$

*where* $\mathrm{C}_{\mathrm{z}}(X) := X \wedge_Z I_Z$*, and in particular the (reduced) suspension*

$$\Sigma_{\mathrm{z}}(X) := X \wedge_Z S_Z^1$$

*of an object* $X \in \mathbf{C}_Z$*.*

***B.4*** *the pushout defining the functor* $\beta_* : \mathbf{C}_Z \to \mathbf{C}_{Z'}$ *on an object* $X \in \mathbf{C}_Z$*,*

$$\begin{tikzcd} Z \arrow[r, "\beta"] \arrow[d, "i_X"'] \arrow[dr, phantom, "\ulcorner"] & Z' \arrow[d] \\ X \arrow[r] & \beta_* X. \end{tikzcd}$$

*Then the tensor-triangulated Spanier–Whitehead category* $\mathbf{SW}(\mathbf{C}_Z)$ *can be constructed as in Chapter 3.*

*Proof.* This follows from a careful examination of each of the individual pushouts used in the construction of $\mathbf{SW}(\mathbf{C}_Z)$ in Chapter 3. □

## 4.2 The Spanier–Whitehead category $\mathbf{SW}(\mathbf{CW}_*)$

The first category we want to construct via the axiomatic approach is the (classical) CW Spanier–Whitehead category $\mathbf{SW}(\mathbf{CW}_*)$ which is a well-known triangulated category; originally defined by Spanier and Whitehead in [SW62], it was studied extensively in [Mar83], and presented here in Chapter 2. Since the category **CW** in particular does not have all finite colimits, it fails to satisfy ambient category conditions **P.1** and **P.2**. However, it does have products and pushouts along closed subcomplexes which commute. We can therefore use the set of assumptions as described in Proposition 4.1.1 in order to construct $\mathbf{SW}(\mathbf{CW}_*)$.

In this section we verify that **CW** together with the object $* \in \mathbf{CW}$ satisfy the weakened set of assumptions, namely **P.3**, **P.4**, **P.5** and **A.1**, **A.2**, **A.3**, **B.1**, **B.2**, **B.3**, **B.4**. We then apply the axiomatic method of Chapter 3 to advance a simple proof that $\mathbf{SW}(\mathbf{CW}_*)$ is a tensor-triangulated Spanier–Whitehead category which includes a clear verification of the octahedral axiom **TR.6**.

**4.2.1 Definition.** The **category of finite CW-complexes, CW,** has objects the finite CW-complexes, and morphisms the continuous cellular maps.

Consider the slice-coslice category relative to the object $* \in \mathbf{CW}$,

$$\mathbf{CW}_* := *\backslash(\mathbf{CW}/*).$$

This is exactly the category of finite pointed CW-complexes.

**4.2.2 Definition.** The **category of finite pointed CW-complexes $\mathbf{CW}_*$** has objects of the form $* \xrightarrow{i_X} \mathbf{X} \xrightarrow{p_X} *$, and a morphism in $\mathbf{CW}_*$ from $* \xrightarrow{i_X} \mathbf{X} \xrightarrow{p_X} *$ to $* \xrightarrow{i_{X'}} \mathbf{X'} \xrightarrow{p_{X'}} *$ is a map $h$ such that the following diagram commutes

$$\begin{tikzcd} & * \arrow[dl, "i_X"'] \arrow[dr, "i_{X'}"] & \\ \mathbf{X} \arrow[rr, "\mathbf{h}"] \arrow[dr, "p_X"'] & & \mathbf{X'}, \arrow[dl, "p_{X'}"] \\ & * & \end{tikzcd}$$

i.e. morphisms in $\mathbf{CW}_*$ are basepoint-preserving cellular maps.

**4.2.3 Remark.** In fact, taking the slice category of **CW** over $*$ is not necessary here since **CW** is already a category over the object $*$, i.e. $*\backslash(\mathbf{CW}/*) = *\backslash\mathbf{CW}$.

**4.2.4 Lemma.** *The category* **CW** *together with the object* $* \in \mathbf{CW}$ *satisfy properties* ***P.3****,* ***P.4****,* ***P.5****,* ***A.1****,* ***A.2****,* ***A.3****, and* ***B.1****,* ***B.2****,* ***B.3****,* ***B.4****.*

*Proof.* Let $\mathbf{C} = \mathbf{CW}$ and $Z = *$.

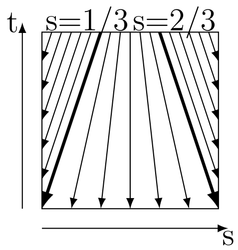
P.3 **CW** has interval object $I = [0,1]$ and 0-sphere object $S^0 = * \vee_\emptyset *$, and there exists a factorisation of the fold morphism $S^0 \to *$ via the morphism $S^0 \to I$:

$$\begin{tikzcd} S^0 \arrow[rr, "id_* \vee_\emptyset id_*"] \arrow[dr, dashed, "c_*"'] & & *. \\ & I \arrow[ur, "\pi_I"'] & \end{tikzcd}$$

P.4 The morphism $c_* : S^0 \to I$ is a cofibration in **CW** since there exists a retraction

$$I \times I \longrightarrow \mathrm{Cyl}(c_*)$$

which can be visualised in the illustration below:




P.5 The transposition morphism $\tau : S^0 \to S^0$ in **CW** extends to the following commutative square so that $\tau^2 = id$ in **CW**

$$\begin{tikzcd} S^0 \arrow[r, "c_*"] \arrow[d, "\tau"] & I \arrow[d, dashed, "\tau"] \\ S^0 \arrow[r, "c_*"] & I \end{tikzcd}$$

where $\tau(t) = 1 - t$.

A.1 Given $f : X \to Y$ in **CW**, since $X \to X \times I$ is a closed subcomplex in **CW**, the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id\times\iota_0"'] & Y \arrow[d] \\ X \times I \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{Cyl}(f) \end{tikzcd}$$

exists in **CW** and distributes over products.

A.2 Given $f : X \to Y$ in **CW**, since $X \to C(X)$ is a closed subcomplex in **CW** where $C(X) := \mathrm{Cyl}(X \xrightarrow{\pi_X} *)$, the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id\times\iota_1"'] & Y \arrow[d] \\ C(X) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C(f) \end{tikzcd}$$

exists in **CW** and distributes over products.

A.3 Since $* \to I$ is a closed subcomplex in **CW**, the pushout

$$\begin{tikzcd} * \arrow[r, "\iota_0"] \arrow[d, "\iota_1"'] & I \arrow[d] \\ I \arrow[r] & I'; \end{tikzcd}$$

exists in **CW** and distributes over products.

B.1 Since $X \vee_* Y \to X \vee_* Y$ is a closed subcomplex in $\mathbf{CW}_*$, the pushout

$$\begin{tikzcd} X \vee_* Y \arrow[r] \arrow[d] & X \times_* Y \arrow[d] \\ * \arrow[r] & X \wedge_* Y; \end{tikzcd}$$

exists in $\mathbf{CW}_*$ and distributes over products.

B.2 Given $f : X \to Y$, since $X \to X \wedge_* I_*^+$ is a closed subcomplex in $\mathbf{CW}_*$, the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id\wedge_* v_1"'] & Y \arrow[d] \\ X \wedge_* I_*^+ \arrow[r] & \mathrm{Cyl}_*(f) \end{tikzcd}$$

exists in $\mathbf{CW}_*$ and distributes over products.

B.3 Given $f : X \to Y$, since $X \to X \wedge_* I_*$ is a closed subcomplex in $\mathbf{CW}_*$, the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id\wedge_* c_1"'] & Y \arrow[d] \\ X \wedge_* I_* \arrow[r] & C_*(f) \end{tikzcd}$$

exists in $\mathbf{CW}_*$ and distributes over products.

B.4 Given a morphism $\beta : * \to Z'$ in **CW** where $Z'$ is some object in **C** satisfying properties B.1–B.3, since $* \to X$ is a closed subcomplex in $\mathbf{CW}_*$, the pushout

$$\begin{tikzcd} * \arrow[r, "\beta"] \arrow[d, "i_X"'] & Z' \arrow[d] \\ X \arrow[r] & \beta_* X \end{tikzcd}$$

exists in $\mathbf{CW}_*$ and distributes over products.

□

Thus, by Proposition 4.1.1, **CW** is an ambient category which yields a tensor-triangulated Spanier–Whitehead category $\mathbf{SW}(\mathbf{CW}_*)$. This will be exactly the classical Spanier–Whitehead category discussed in Chapter 2 (as defined and studied in [Mar83]). However, for completeness, we give the definition of the category $\mathbf{SW}(\mathbf{CW}_*)$ below.

**4.2.5 Definition.** The **(classical) CW Spanier–Whitehead category**, denoted $\mathbf{SW}(\mathbf{CW}_*)$, has as objects pairs $(X, m)$, where $X \in \mathbf{CW}_*$ and $m \in \mathbb{Z}$, and morphisms

$$\mathrm{Hom}_{\mathbf{SW}(\mathbf{CW}_*)}\big((X,m),(Y,n)\big) := \mathrm{colim}_{k\to\infty}[\varSigma_*^{k+m}(X), \varSigma_*^{k+n}(Y)]$$

(where $[P, Q]$ denotes the set of homotopy classes of morphisms $P \to Q$ in $\mathbf{CW}_*$).

Now the well-known theorem below follows immediately from the axiomatic approach to Spanier–Whitehead categories described in Chapter 3.

**4.2.6 Theorem.** $\mathbf{SW}(\mathbf{CW}_*)$ *is a tensor triangulated category.*

*Proof.* Letting $\mathbf{C} = \mathbf{CW}$ and $Z = *$, this follows from Lemma 3.5.10. □

**4.2.7 Remark.** This alternative method for showing that $\mathbf{SW}(\mathbf{CW}_*)$ is triangulated provides a clear proof that $\mathbf{SW}(\mathbf{CW}_*)$ satisfies the octahedral axiom, **TR.6**.

Recall that any morphism in $\mathbf{SW}(\mathbf{CW}_*)$ can be represented by a cofibration in $\mathbf{CW}_*$ by Lemma 3.5.4. In particular, given a triangle

$$X \xrightarrow{f} Y \to C_*(f) \to \varSigma_*(X) \to \ldots$$

in $\mathbf{SW}(\mathbf{CW}_*)$, we may assume that $f : X \to Y$ is a cofibration. Then, by Lemma 3.3.7, $C_*(X \to Y) \simeq Y/X$ in $\mathbf{CW}_*$. Hence $C_*(X \to Y) \cong Y/X$ in $\mathbf{SW}(\mathbf{CW}_*)$, and the above triangle is isomorphic to

$$X \xrightarrow{f} Y \to Y/X \to \varSigma_*(X) \to \ldots .$$

**4.2.8 Corollary. The Grothendieck group of $\mathbf{SW}(\mathbf{CW}_*)$,** *denoted by*

$$K(\mathbf{SW}(\mathbf{CW}_*)),$$

*is generated by the classes $[X]$ for $X \in \mathbf{CW}_*$ subject to the relations*

$$[Y] = [X] + [Y/X]$$

*whenever there is a cofibration $X \to Y$.*

**4.2.9 Lemma.** $K(\mathbf{SW}(\mathbf{CW}_*))$ *is a ring.*

*Proof.* By Corollary 3.5.11, setting $\mathbf{C} = \mathbf{CW}$ and $Z = *$. □

Following the same reasoning as will be used in proving Theorem 4.5.6, we expect the ring $K(\mathbf{SW}(\mathbf{CW}_*))$ to be isomorphic to the integers. We note that, if $Z$ is fixed to be some higher dimensional object in **CW** other than the point $*$, the set of weakened assumptions of Proposition 4.1.1 do not hold (except possibly for the case where $Z$ is a disjoint union of points). We are therefore unable to construct a relative version of the CW Spanier–Whitehead category over some fixed object $Z \in \mathbf{CW}$ (unless $Z = *$). In order to construct relative Spanier–Whitehead categories $\mathbf{SW}(\mathbf{C}_Z)$ over fixed $Z \in \mathbf{C}$, we need to start with an ambient category **C** which consists of tamer spaces. This is the motivation for considering the category of pointed compact definable spaces **Def** as an ambient category, and it will become apparent in Section 4.6 that **Def** is indeed the ideal candidate for such relative Spanier–Whitehead category constructions.

## 4.3 The categories Def and $\mathbf{Def}_Z$

We now fix an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$ so that definable means $\mathcal{R}$-definable, and consider the category of compact definable spaces **Def**.

In order to construct the definable Spanier–Whitehead category, $\mathbf{SW}(\mathbf{Def}_*)$, and the relative definable Spanier–Whitehead category, $\mathbf{SW}(\mathbf{Def}_Z)$, we first need to verify that **Def** is a satisfactory choice of ambient category. Again it is doubtful whether **Def** has all finite colimits. The following is an example illustrating the matter in question.

**4.3.1 Example.** Consider the map $f : S^1 \to S^1$ given in polar coordinates by $f(\theta) = \theta + \phi$ where $\phi$ is an irrational multiple of $\pi$. This is definable since in cartesian coordinates it is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}.$$

Consider the coequaliser of

$$S^1 \underset{\mathrm{f}}{\overset{\mathrm{id}}{\rightrightarrows}} S^1.$$

This is the quotient of $S^1$ by the equivalence relation $E \subset S^1 \times S^1$ generated by

$(\theta, f(\theta)) = (\theta, \theta + \phi)$ for $\theta \in S^1$. This is easily seen to be

$$E = \{(\theta, \theta + n\phi) \mid \theta \in S^1, n \in \mathbb{Z}\}.$$

However, this is neither definable nor closed. For instance, the fibre of $p_1 : E \to S^1$ over $\theta \in S^1$ is $\{\theta + n\phi \mid n \in \mathbb{Z}\}$ which is dense (but not closed) and is not definable.

This example shows that either some coequalisers do not exist in **Def**, or they exist but do not 'forget' to the coequaliser in **Top** (this would be the case if the generated equivalence relation in the example above could be shown to sit inside some minimal closed definable equivalence relation). In any case it is unclear whether or not **Def** has finite colimits, and hence whether or not **Def** satisfies **P.1** and **P.2**.

However, by Lemma 1.3.3 we know that if $i : A \to B$ in **Def** is a closed inclusion, then the pushout along $i$ and some given morphism $f : A \to C$ exists. Thus we can prove that **Def** satisfies the set of weaker assumptions given in Proposition 4.1.1 for any $Z \in$ **Def**. We will then be able to construct the Spanier–Whitehead categories of the slice-coslice category $\mathbf{Def}_Z$ which we know will be tensor-triangulated categories.

This section consists of the verification that **Def** satisfies the conditions **P.3**, **P.4**, **P.5**, **A.1**, **A.2**, **A.3**, and that for any $Z \in$ **Def**, conditions **B.1**, **B.2**, **B.3**, **B.4** hold.

**4.3.2 Definition.** The **definable category, Def,** has objects the compact definable subsets (in an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$), and morphisms the continuous definable maps.

Consider the definable slice-coslice category relative to an object $Z \in$ **Def**,

$$\mathbf{Def}_Z := Z \backslash (\mathbf{Def}/Z).$$

This is exactly the category of relative definable spaces.

**4.3.3 Definition.** The **relative definable category, $\mathbf{Def}_Z$,** has objects of the form $Z \xrightarrow{i_X} \mathbf{X} \xrightarrow{p_X} Z$, i.e. relative compact definable subsets (in an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$). A morphism in $\mathbf{Def}_Z$ from $Z \xrightarrow{i_X} \mathbf{X} \xrightarrow{p_X} Z$ to $Z \xrightarrow{i_{X'}} \mathbf{X}' \xrightarrow{p_{X'}} Z$

is a map $h$ such that the following diagram commutes

$$\begin{tikzcd}
 & Z \arrow[dl, "i_X"'] \arrow[dr, "i_{X'}"] & \\
\mathbf{X} \arrow[rr, "\mathbf{h}"] \arrow[dr, "p_X"'] & & \mathbf{X}', \arrow[dl, "p_{X'}"] \\
 & Z &
\end{tikzcd}$$

i.e. morphisms in $\mathbf{Def}_Z$ are relative continuous definable maps.

**4.3.4 Lemma.** *The category* **Def** *together with the object* $Z \in$ **Def** *satisfy properties* ***P.3****,* ***P.4****,* ***P.5****,* ***A.1****,* ***A.2****,* ***A.3****, and* ***B.1****,* ***B.2****,* ***B.3****,* ***B.4****.*

*Proof.* Let $\mathbf{C} = \mathbf{Def}$ and suppose $Z \in \mathbf{Def}$.

P.3 **Def** has interval object $I = [0,1]$ and 0-sphere object $S^0 = * \vee_\emptyset *$, and there exists a factorisation of the fold morphism $S^0 \to *$ via the morphism $S^0 \to I$:

$$\begin{tikzcd}
S^0 \arrow[rr, "id_* \vee_\emptyset id_*"] \arrow[dr, dashed, "c_*"'] & & *. \\
 & I \arrow[ur, "\pi_I"'] &
\end{tikzcd}$$

P.4 The morphism $c_* : S^0 \to I$ is a cofibration in **Def** since there exists a retraction

$$I \times I \longrightarrow \mathrm{Cyl}(c_*)$$

which can be visualised as illustrated in the proof of Lemma 4.2.4.

P.5 The transposition morphism $\tau : S^0 \to S^0$ in **Def** extends to the following commutative square so that $\tau^2 = id$ in **Def**.

$$\begin{tikzcd}
S^0 \arrow[r, "c_*"] \arrow[d, "\tau"] & I \arrow[d, dashed, "\tau"] \\
S^0 \arrow[r, "c_*"] & I
\end{tikzcd}$$

where we define $\tau(t) = 1 - t$.

Properties **A.1**, **A.2**, **A.3**, and **B.1**, **B.2**, **B.3**, **B.4** follow from Lemma 1.3.3 which tells us that if $A \to B$ in **Def** is a closed inclusion, then given some morphism $f : A \to C$ in **Def** the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & C \arrow[d] \\ B \arrow[r] & B \vee_f C \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

exists in **Def**, and from the fact that products commute with pushouts in **Def**. □

Thus, by Proposition 4.1.1, **Def** is an ambient category which yields tensor-triangulated Spanier–Whitehead categories for any $Z \in \mathbf{Def}$.

## 4.4 The definable Spanier–Whitehead category, $\mathbf{SW}(\mathbf{Def}_*)$

Fixing an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$, considering the ambient category **Def** and the case where $Z = *$, we now focus on the construction of the (absolute) definable Spanier–Whitehead category $\mathbf{SW}(\mathbf{Def}_*)$. Using the axiomatic approach of Chapter 3, the fact that $\mathbf{SW}(\mathbf{Def}_*)$ is a tensor-triangulated category will follow immediately. Given a morphism $f : X \to Y$ in the definable Spanier–Whitehead category, a triangle will be of the form $X \xrightarrow{f} Y \to C_{\mathbf{Def}_*}(f) \to \Sigma_{\mathbf{Def}_*}(X) \to ...$ where $C_{\mathbf{Def}_*}(f) \cong Y/X$ in $\mathbf{SW}(\mathbf{Def}_*)$ since there is a homotopy equivalence in $\mathbf{Def}_*$. The Grothendieck group is then generated by the classes $[X]$ subject to the relation $[Y] = [X]+[Y/X]$ given by triangles, and is a ring. The Grothendieck group of $\mathbf{SW}(\mathbf{Def}_*)$ will be shown to be isomorphic to the integers.

Consider the slice-coslice category of **Def** relative to $*$, also known as the pointed definable category, $\mathbf{Def}_* := *\backslash(\mathbf{Def}/*)$, as defined below.

**4.4.1 Definition.** The **pointed definable category, $\mathbf{Def}_*$,** has objects of the form $* \xrightarrow{i_X} \mathbf{X} \xrightarrow{p_X} *$, i.e. pointed compact definable subsets (in an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$), and morphisms are basepoint-preserving continuous definable maps.

We note that, in fact, taking the slice category of **Def** over $*$ is not required here since **Def** is already a category over the object $*$, i.e. $*\backslash(\mathbf{Def}/*) = *\backslash\mathbf{Def}$. We can now give the definition of the definable Spanier–Whitehead category.

**4.4.2 Definition. The definable Spanier–Whitehead category**, denoted $\mathbf{SW}(\mathbf{Def}_*)$, has as objects pairs $(X, m)$, where $X \in \mathbf{Def}_*$ and $m \in \mathbb{Z}$, and morphisms

$$\mathrm{Hom}_{\mathbf{SW}(\mathbf{Def}_*)}\big((X,m),(Y,n)\big) := \mathrm{colim}_{k\to\infty}[\Sigma^{k+m}_{\mathbf{Def}_*}(X), \Sigma^{k+n}_{\mathbf{Def}_*}(Y)]^{\mathcal{R}}$$

(where $[P,Q]^{\mathcal{R}}$ denotes the set of definable homotopy classes of morphisms $P \to Q$ in $\mathbf{Def}_*$).

**4.4.3 Theorem.** $\mathbf{SW}(\mathbf{Def}_*)$ *is a tensor triangulated category.*

*Proof.* Letting $\mathbf{C} = \mathbf{Def}$ and $Z = *$, this follows from Lemma 3.5.10. $\square$

Recall that the objects in the standard Spanier–Whitehead category $\mathbf{SW}(\mathbf{CW}_*)$ are the pairs $(X, m)$, where $X$ is a finite pointed CW-complex and $m \in \mathbb{Z}$, and that the morphisms are the pointed homotopy classes of continuous basepoint preserving maps,

$$\mathrm{Hom}_{\mathbf{SW}(\mathbf{CW}_*)}\big((X,m),(Y,n)\big) := \mathrm{colim}_{k\to\infty}[\Sigma^{k+m}(X), \Sigma^{k+n}(Y)].$$

The definable Spanier–Whitehead category $\mathbf{SW}(\mathbf{Def}_*)$ is defined just as $\mathbf{SW}(\mathbf{CW}_*)$ but with compact definable spaces in place of finite CW-complexes, and with, definable suspensions and definable homotopies.

**4.4.4 Definition.** The **underlying topology functor** $U$ from $\mathbf{SW}(\mathbf{Def}_*)$ to $\mathbf{SW}(\mathbf{CW}_*)$ picks out the underlying spaces and underlying continuous maps. Formally:

$$U : \mathbf{SW}(\mathbf{Def}_*) \to \mathbf{SW}(\mathbf{CW}_*)$$

is defined by

(i) associating to each object $(X, m)$ in $\mathbf{SW}(\mathbf{Def}_*)$, the pair $(U(X), m)$ where $U(X)$ is the underlying topological space of $X$, i.e. $U((X, m)) = (U(X), m)$.

(ii) taking a morphism $\mathrm{colim}_{k\to\infty}[\Sigma^{k+m}_{\mathbf{Def}_*}(X), \Sigma^{k+n}_{\mathbf{Def}_*}(Y)]^{\mathcal{R}}$ to

$$\mathrm{colim}_{k\to\infty}[\Sigma^{k+m}(U(X)), \Sigma^{k+n}(U(Y))]$$

(since a continuous definable map is in particular a continuous map, and we know by Lemma 1.3.26 that if two maps are definably homotopic then they are also homotopic in the classical sense, and by Corollary 1.3.22 that $U(\Sigma_{\mathbf{Def}_*}(X)) \cong \Sigma(U(X))$).

The following result gives a comparison between the semalgebraic-definable Spanier–Whitehead category (where the fixed o-minimal expansion of $\mathbb{R}$ is taken to be the semialgebraic subsets of $\mathbb{R}^n$) and the classical Spanier–Whitehead category of CW-complexes.

**4.4.5 Proposition.** *Fix* $\mathcal{R}$ *to be the semialgebraic o-minimal expansion* $\mathcal{R}^{\mathcal{SA}}$. *Then* $U$ *is a fully-faithful functor whose image is a full triangulated subcategory of* $\mathbf{SW}(\mathbf{CW}_*)$.

*Proof.* This is a direct corollary of Theorem 1.3.34 (and Corollary 1.3.35) in Section 1.3.24. The image of $G$ is closed under suspensions and closed under taking mapping cones of morphisms (by Corollary 1.3.22). Thus the image of $G$ is a full triangulated subcategory of $\mathbf{SW}(\mathbf{CW}_*)$. $\square$

We note that since this result holds for the semialgebraic subsets of $\mathbb{R}^n$ which is smallest, and hence least flexible, o-minimal expansion of $\mathbb{R}$, we would expect that an analogous comparison for any $\mathcal{R}$-definable Spanier–Whitehead category also holds.

## 4.5 The Grothendieck group of $\mathbf{SW}(\mathbf{Def}_*)$

We now want to compute the Grothendieck Group of the definable Spanier Whitehead category. From the axiomatic approach to Spanier–Whitehead categories in Chapter 3, we know that $K(\mathbf{SW}(\mathbf{Def}_*))$ is generated by classes of objects in $\mathbf{Def}_*$ subject to relations given by triangles in $\mathbf{SW}(\mathbf{Def}_*)$, and that it has the structure of a ring where the sum and product are induced from the wedge sum $\vee_Z$ and smash product $\wedge_Z$. In this section we prove that, since $\mathbf{Def}$ is a category of tame spaces and because every compact definable space has a CW-structure (via a well-based cylindrical definable cell decomposition as in Section 1.2 of Chapter 1), the ring $K(\mathbf{SW}(\mathbf{Def}_*))$ is actually generated by the class of the 0-sphere $S^0_* \in \mathbf{Def}_*$. We will then be able to prove a key theorem that tells us that $K(\mathbf{SW}(\mathbf{Def}_*))$ is isomorphic to the integers.

Recall that an object $(X, m)$ in $\mathbf{SW}(\mathbf{Def}_*)$, where $m \geq 0$, can be thought of as the object $(\Sigma^m_{\mathbf{Def}_*}(X), 0)$. In this section we denote an object $(X, 0)$ in $\mathbf{SW}(\mathbf{Def}_*)$ simply by $X$. A triangle in $\mathbf{SW}(\mathbf{Def}_*)$ is isomorphic after (de)-suspending to a Puppe sequence

$$X \xrightarrow{f} Y \to C_{\mathbf{Def}_*}(f) \to \Sigma_{\mathbf{Def}_*}(X) \to \dots .$$

Since any morphism in $\mathbf{SW}(\mathbf{Def}_*)$ can be represented by a cofibration by Lemma 3.5.4, in particular we may assume that $f : X \to Y$ is a cofibration. Then, by Lemma 3.3.7, $C_{\mathbf{Def}_*}(X \to Y) \simeq Y/X$ in $\mathbf{Def}_*$. Hence $C_{\mathbf{Def}_*}(X \to Y) \cong Y/X$ in $\mathbf{SW}(\mathbf{Def}_*)$.

**4.5.1 Corollary. The Grothendieck group of $\mathbf{SW}(\mathbf{Def}_*)$**, *denoted by*

$$K(\mathbf{SW}(\mathbf{Def}_*)),$$

*is generated by the classes $[X]$ for $X \in \mathbf{Def}_*$ subject to the relations*

$$[Y] = [X] + [Y/X]$$

*whenever there is a cofibration $X \to Y$.*

**4.5.2 Lemma.** *$K(\mathbf{SW}(\mathbf{Def}_*))$ is a ring.*

*Proof.* By Corollary 3.5.11, setting $\mathbf{C} = \mathbf{Def}$ and $Z = *$. □

We now want to prove that the classes in the Grothendieck group $K(\mathbf{SW}(\mathbf{Def}_*))$ are generated by classes of the 0-sphere in $\mathbf{Def}_*$.

Topologically speaking, *a (pointed) sphere* is considered to be any space which is homeomorphic to the standard sphere, $S^n$ in Euclidean space, i.e. the space given by the equation $x_0^2 + x_1^2 + ... + x_n^2 = 1 \subset \mathbb{R}^n$. So in particular, a cube with its boundary collapsed, $I^n/\partial I^n$, is a topological sphere. We will now show that in a definable setting we are able to continue thinking about spheres in this manner and that the two notions of spheres mentioned above are isomorphic in the definable Spanier–Whitehead category.

**4.5.3 Lemma.** *$I_*^n/\partial I_*^n \cong S_*^n$ in $\mathbf{SW}(\mathbf{Def}_*)$.*

*Proof.* By Lemma 3.2.13 we know that $S_*^n := \Sigma^n_{\mathbf{Def}_*}(S_*^0)$. In particular

$$S_*^n \cong \Sigma_{\mathbf{Def}_*}(S_*^{n-1})$$

. Thus we need to prove that $I_*^n/\partial I_*^n \cong \Sigma_{\mathbf{Def}_*}(S_*^{n-1})$ in $\mathbf{SW}(\mathbf{Def}_*)$. Consider the following triangle in $\mathbf{SW}(\mathbf{Def}_*)$:

$$\partial I_*^n \xrightarrow{f} I_*^n \to I_*^n/\partial I_*^n \to \Sigma_{\mathbf{Def}_*}(\partial I_*^n) \to ...$$

where $I_*^n/\partial I_*^n \cong C_{\mathbf{Def}_*}f$. Since $I_*^0 \simeq *$ and $I_*^n \simeq *$ in $\mathbf{Def}_*$, we have $I_*^0 \cong I_*^n$ in $\mathbf{SW}(\mathbf{Def}_*)$. So the above triangle is isomorphic to

$$\partial I_*^n \to 0 \to I_*^n/\partial I_*^n \to \Sigma_{\mathbf{Def}_*}(\partial I_*^n) \to 0.$$

Thus, in $\mathbf{SW}(\mathbf{Def}_*)$ we have

$$I_*^n/\partial I_*^n \cong \Sigma_{\mathbf{Def}_*}(\partial I_*^n).$$

There is a definable homeomorphism $S_*^{n-1} \to \partial I_*^n$ (the obvious radial projection) in $\mathbf{Def}_*$. So in $\mathbf{SW}(\mathbf{Def}_*)$ there is an isomorphism $S_*^{n-1} \cong \partial I_*^n$. Hence

$$I_*^n/\partial I_*^n \cong \Sigma_{\mathbf{Def}_*}(S_*^{n-1}) \tag{4.5.1}$$

in the definable Spanier–Whitehead category, $\mathbf{SW}(\mathbf{Def}_*)$. In other words, an $n$-dimensional cube with its boundary collapsed is isomorphic to the *genuine* $n$-sphere, $S_*^n := \Sigma_{\mathbf{Def}_*}(S_*^{n-1})$. □

**4.5.4 Lemma.**
$$[S^n_*] = -[S^{n-1}_*] = (-1)^n[S^0_*]$$

*Proof.* For a closed interval $[0,1]$ (with basepoint $*$ at the 0-end), we have:

$$[[0,1]] = [S^0_*] + [S^1_*].$$

We can also think of the closed interval as the union of the first half and the second half (the whole interval with the first half collapsed down):

$$[[0,1]] = [[0,1/2]] + [[0,1]/[0,1/2]] = [[0,1/2]] + [[1/2,1]].$$

So this is equivalent to
$$[[0,1]] = [[0,1]] + [[0,1]]].$$

Thus we obtain $[[0,1]] = 0$, so that

$$[S^0_*] + [S^1_*] = [[0,1]] = 0,$$

i.e.
$$[S^1_*] = -[S^0_*].$$

Let us suppose that $[S^n_*] = (-1)^n[S^0_*]$. Then

$$[S^{n+1}_*] = [S^n_*] + [S^{n+1}_*/S^n_*] = [S^n_*] + [S^{n+1}_* \vee_* S^{n+1}_*].$$

In general we have that

$$[A \vee_* B] = [A] + [A \vee_* B/A] = [A] + [B].$$

So $[S^{n+1}_*] = [S^n_*] + [S^{n+1}_*] + [S^{n+1}_*]$. Thus

$$\begin{aligned}[S^{n+1}_*] &= -[S^n_*]\\ &= -(-1)^n[S^0_*]\\ &= (-1)^{n+1}[S^0_*].\end{aligned}$$

□

The following lemma is required in order to prove that $K(\mathbf{SW}(\mathbf{Def}_*)) \cong \mathbb{Z}$ in Theorem 4.5.6 below. We note that since the objects in $\mathbf{Def}_*$ are definable spaces which are compact, we can consider the Euler characteristic $\chi$ (rather than the compactly-supported Euler characteristic $\chi_C$).

**4.5.5 Lemma.** *Suppose* $(X,m) \in \mathbf{SW}(\mathbf{Def}_*)$. *Then*

$$[(X,m)] = (-1)^m(\chi(X)-1)[S^0_*].$$

*Proof.* We can choose a well-based definable cell decomposition of $Y$ by Theorem 1.2.16 (so that, in addition, we have that the closure of a definable cell is homeomorphic to a standard cell). We take $[Y_0]$ to be the 0-skeleton, $[Y_1]$ the 1-skeleton, etc. Then, inductively from the relation $[Y]=[X]+[Y/X]$ we obtain:

$$[Y] = [Y_0] + [Y_1/Y_0] + [Y_2/Y_1] + ... + [Y_n/Y_{n-1}] \tag{4.5.2}$$

where $[Y_0]$ is a wedge of 0-spheres, $[Y_1/Y_0]$ is a wedge of 1-spheres, etc. If in a cell decomposition the number of 0-cells is $k$, then we consider the 0-cells as a wedge of $(k-1)$ 0-spheres:

$$\underbrace{\bullet ... \bullet}_{\text{k}} = \vee_{k-1} S^0_*.$$

Then, by Lemma 4.5.4, each term in equation (4.5.2) is generated by (wedges of) 0-spheres:

$$[Y] = \sum_{\dim C>0} ([C/\partial C]) + (\#0\text{-cells} - 1)[S^0_*]$$

where the sum is taken over the dimensions of the cells $C$ in the well-based cell decomposition of $Y$.

Since the cell decomposition is well-based, we have one CW-cell for each definable cell in the decomposition. So, for $\dim C > 0$, by Lemma 4.5.3, we have

$$[C/\partial C] = [S^{\dim C}_*],$$

hence

$$[Y] = \sum_{\dim C>0} (-1)^{\dim C}[S^0_*] + \sum_{\dim C=0} [S^0_*] - [S^0_8] = \sum_{\dim C\geq 0} (-1)^{\dim C}[S^0_*] - [S^0_*].$$

Therefore

$$[Y] = (\chi(Y)-1)\cdot[S^0_*].$$

In the definable Spanier–Whitehead category, given a pair $(X,m) \in \mathbf{SW}(\mathbf{Def}_*)$, we have:

$$(X,m) \cong (\varSigma_{\text{Def}_*}(X), m-1) \cong T(X,m-1) \cong T^m(X,0)$$

where, here, $T$ denotes the shift functor given by $T(X,m) = (X, m+1)$. Thus

$$[(X,m)] = [(X,0)](-1)^m = (-1)^m(\chi(X) - 1)[S^0_*].$$

□

Now, with the knowledge that the ring $K(\mathbf{SW}(\mathbf{Def}_*))$ is generated by the class of $S^0_*$ as in the lemma above, we can state and prove the following important result.

**4.5.6 Theorem.** *There exists a ring isomorphism*

$$\tilde{\chi} : K(\mathbf{SW}(\mathbf{Def}_*)) \to \mathbb{Z}$$

*given by*

$$[(X,m)] \mapsto (-1)^m \tilde{\chi}(X) \tag{4.5.3}$$

*where* $\tilde{\chi}(X) = \chi(X) - 1$.

*Proof.* First we note that, since the Grothendieck group is generated by the classes of $[(X,0)]$ (by Lemma 4.5.5), we can reduce our considerations to the case where $m = n = 0$. The Euler characteristic $\tilde{\chi}$, which maps spaces to the integers, descends to the Grothendieck group, i.e. $\tilde{\chi}([(X,0)]) := \tilde{\chi}(X)$. Hence the map (4.5.3) is well-defined. The product in the definable Spanier–Whitehead category is given by

$$(X,m) \wedge_* (Y,n) := (X \wedge_* Y, m+n).$$

So we need to prove that

$$\tilde{\chi}[(X,0) \vee_* (Y,0)] = \tilde{\chi}[(X,0)] + \tilde{\chi}[(Y,0)]$$

(which is clear), and that

$$\tilde{\chi}[(X,0) \wedge_* (Y,0)] = \tilde{\chi}[(X,0)] \cdot \tilde{\chi}[(Y,0)], \tag{4.5.4}$$

where, by Lemma 4.5.5, $[(X,0)] = (\chi(X) - 1)[S^0_*]$ and $[(Y,0)] = (\chi(Y) - 1)[S^0_*]$, and

$$[(X,0) \wedge_* (Y,0)] = [(X \wedge_* Y, 0)] = (\chi(X \wedge_* Y) - 1)[S^0_*].$$

Hence

$$\tilde{\chi}[(X,0) \wedge_* (Y,0)] = \tilde{\chi}[(X \wedge_* Y, 0)] = \tilde{\chi}(X \wedge Y),$$

and

$$\tilde{\chi}[(X,0)] \cdot \tilde{\chi}[(Y,0)] = \tilde{\chi}(X) \cdot \tilde{\chi}(Y).$$

So proving (4.5.4) can be reduced to showing that

$$\tilde{\chi}(X \wedge_* Y) = \tilde{\chi}(X) \cdot \tilde{\chi}(Y).$$

Recalling that $\tilde{\chi}(A) = \chi(A) - 1$, we rewrite this expression as:

$$\chi(X \wedge_* Y) - 1 = (\chi(X) - 1) \cdot (\chi(Y) - 1) = \chi(X) \cdot \chi(Y) - \chi(X) - \chi(Y) + 1.$$

The smash product of $X$ and $Y$ is defined as $X \wedge_* Y = (X \times_* Y)/X \vee_* Y$. Given $B \subset A$ in $\mathbf{Def}_*$, we know that

$$\chi(A/B) = \chi(A) - \chi(B) + 1$$

and $\chi(A \times_* B) = \chi(A) \cdot \chi(B)$. The wedge sum of $X$ and $Y$ is the quotient space of the disjoint union $X \vee_\emptyset Y$ where the basepoints of $X$ and $Y$ are identified. So $\chi(X \vee_* Y) = \chi(X) + \chi(Y) - 2 \cdot \chi(*) + 1$ Hence

$$\begin{aligned}
\chi(X \wedge_* Y) &= \chi(X \times_* Y) - \chi(X \vee_* Y) + 1 \\
&= \chi(X) \cdot \chi(Y) - (\chi(X) + \chi(Y) - 2 \cdot \chi(*) + 1) + 1 \\
&= \chi(X) \cdot \chi(Y) - \chi(X) - \chi(Y) + 2.
\end{aligned}$$

Subtracting 1 from both sides, we obtain

$$\chi(X \wedge_* Y) - 1 = \chi(X) \cdot \chi(Y) - \chi(X) - \chi(Y) + 1,$$

i.e. in terms of the reduced Euler characteristic,

$$\tilde{\chi}(X \wedge_* Y) = \tilde{\chi}(X) \cdot \tilde{\chi}(Y), \tag{4.5.5}$$

as required. Hence $[(X \wedge_* Y, m+n)] = [(X, m)] \cdot [(Y, n)]$. In the definable Spanier–Whitehead category, $\mathbf{SW}(\mathbf{Def}_*)$, the unit is $(S^0_*, 0)$ since

$$(S^0_*, 0) \wedge_* (X, m) = (S^0_* \wedge_* X, m) = (X, m).$$

Then $[(S^0_*, 0)] = (-1)^0(\chi(S^0_*) - 1)[S^0_*] = (2 - 1)[S^0_*] = [S^0_*]$, by Lemma 4.5.5. So $\tilde{\chi}[(S^0_*, 0)] = \tilde{\chi}(S^0_*) = 1$. Therefore $\tilde{\chi} : K(\mathbf{SW}(\mathbf{Def}_*)) \to \mathbb{Z}$, given by $\tilde{\chi}[(X, m)] = (-1)^m \tilde{\chi}(X)$, is surjective. Since $\tilde{\chi}[(X, m)] = 0$ if and only if $[(X, m)]$ is 0, it is also injective. Thus $\tilde{\chi}$ is a bijective ring homomorphism, as required. $\square$

## 4.6 The relative definable Spanier–Whitehead category, $\mathbf{SW}(\mathbf{Def}_Z)$

At the core of this thesis is the proof that the Grothendieck group of the relative definable Spanier–Whitehead category is exactly the ring of constructible functions over $Z$, denoted $CF(Z)$.

We recall that in the case where $\mathbf{CW}$ is the ambient category, the set of weaker assumptions of Proposition 4.1.1 hold for $Z = *$, but are unlikely to hold for any other $Z \in \mathbf{CW}$. Hence we are not able to construct a relative version of the CW Spanier–Whitehead category over a fixed object $Z \in \mathbf{CW}$ except over $*$, and possibly over disjoint unions of copies of $*$. In the case of a higher-dimensional object $Z \in \mathbf{CW}$, the morphisms may behave wildly over the base-object $Z$ meaning for instance that we would struggle to consider fibres and pullbacks. This highlights that, in order to construct relative versions of the Spanier–Whitehead construction, there is the need for a tame ambient category in which such issues do not arise. The definable category is precisely such an ambient category.

Fixing an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$, considering the ambient category $\mathbf{Def}$ and some $Z \in \mathbf{Def}$, we can now follow the axiomatic approach of Chapter 3, using the set of weaker assumptions on the ambient category, to construct the relative version of the definable Spanier–Whitehead category. This is defined below, and the proof that is is a tensor-triangulated category follows immediately from Chapter 3 by setting $\mathbf{C} = \mathbf{Def}$ and fixing some $Z \in \mathbf{Def}$.

Consider the definable slice-coslice category relative to an object $Z \in \mathbf{Def}$,

$$\mathbf{Def}_Z := Z\backslash(\mathbf{Def}/Z).$$

**4.6.1 Definition. The relative definable Spanier–Whitehead category**, denoted $\mathbf{SW}(\mathbf{Def}_Z)$, has as objects pairs $(X, m)$, where $X \in \mathbf{Def}_Z$ and $m \in \mathbb{Z}$, and morphisms

$$\mathrm{Hom}_{\mathbf{SW}(\mathbf{Def}_Z)}\big((X,m),(Y,n)\big) := \mathrm{colim}_{k\to\infty}[\Sigma^{k+m}_{\mathbf{Def}_Z}(X), \Sigma^{k+n}_{\mathbf{Def}_Z}(Y)]^{\mathcal{R}}$$

(where $[P, Q]^{\mathcal{R}}$ denotes the set of definable homotopy classes of morphisms $P \to Q$ in $\mathbf{Def}_Z$).

**4.6.2 Theorem.** $\mathbf{SW}(\mathbf{Def}_Z)$ *is a tensor triangulated category.*

*Proof.* By Lemma 4.3.4, the category $\mathbf{Def}$ together with any $Z \in \mathbf{Def}$ satisfy properties **P.3**, **P.4**, **P.5**, **A.1**, **A.2**, **A.3**, and **B.1**, **B.2**, **B.3**, **B.4**. So, by

Proposition 4.1.1, we can apply the axiomatic method of Chapter 3. Therefore, letting $\mathbf{C} = \mathbf{Def}$ and fixing some $Z \in \mathbf{Def}$, it follows from Theorem 3.5.8 and Lemma 3.5.10 that $\mathbf{SW}(\mathbf{Def}_Z)$ is tensor triangulated. □

## 4.7 The Grothendieck group of $\mathbf{SW}(\mathbf{Def}_Z)$

We now want to compute the Grothendieck group of the relative definable Spanier Whitehead category. We begin by defining the Grothendieck group $K(\mathbf{SW}(\mathbf{Def}_Z))$ of the relative definable Spanier Whitehead category and showing that it is a ring; these both follow immediately from the axiomatic approach in Chapter 3. We then put forward and prove the pivotal theorem which states that there exists an isomorphism $\tilde{\chi}_Z$ between $K(\mathbf{SW}(\mathbf{Def}_Z))$ and the ring of constructible functions $CF(Z)$ over $Z$ which is given by $\tilde{\chi}_z([(A \xrightarrow{p_A} Z, m)]) = (z \mapsto (-1)^m \tilde{\chi}(p_A^{-1} z))$. To prove that $\tilde{\chi}_Z$ is a well-defined homomorphism, we use the fact that there exists a relative cell decomposition of $p_A : A \to Z$ (by the Trivialisation Theorem 1.1.20) so that the image under $\tilde{\chi}_Z$ is constructible and the fact that the inclusion $\{z\} \to Z$ induces a triangulated functor $\mathbf{SW}(\mathbf{Def}_Z) \to \mathbf{SW}(\mathbf{Def}_z)$ which descends to the Grothendieck groups, together with Theorem 4.5.6 which tells us that $\tilde{\chi}$ is a well-defined homomorphism. Bijectivity is then proved using that $CF(Z)$ is generated by indicator functions and by showing that the image of a class $[(A \xrightarrow{p_A} Z, 0)]$ under $\tilde{\chi}_Z$ can be written as a weighted sum of indicator functions over the cells in $Z$ where the weights in the sum are given by the reduced Euler characteristic of the fibres over the cells.

Note that an object $(X \xrightarrow{p_X} Z, m)$ in $\mathbf{SW}(\mathbf{Def}_Z)$, where $m \geq 0$, can be thought of as the object $(\Sigma^m_{\mathbf{Def}_Z}(X) \xrightarrow{p} Z, 0)$. In this section we often denote an object $(X \xrightarrow{p_X} Z, 0)$ in $\mathbf{SW}(\mathbf{Def}_Z)$ simply by $X$. A triangle in $\mathbf{SW}(\mathbf{Def}_Z)$ is a sequence

$$X \xrightarrow{f} Y \to C_{\mathbf{Def}_Z}(f) \to \Sigma_{\mathbf{Def}_*}(X) \to \dots .$$

Recall that any morphism in $\mathbf{SW}(\mathbf{Def}_Z)$ can be represented by a cofibration by Lemma 3.5.4. In particular we may assume that $f : X \to Y$ is a cofibration. Then, by Lemma 3.3.7, $C_{\mathbf{Def}_Z}(X \to Y) \simeq Y/X$ in $\mathbf{Def}_Z$. Hence $C_{\mathbf{Def}_Z}(X \to Y) \cong Y/X$ in $\mathbf{SW}(\mathbf{Def}_Z)$.

**4.7.1 Corollary.** *The* **Grothendieck group of $\mathbf{SW}(\mathbf{Def}_Z)$***, denoted by*

$$K(\mathbf{SW}(\mathbf{Def}_Z)),$$

*is generated by the classes $[(X \xrightarrow{p_X} Z, 0)]$ subject to the following relations given* *by the triangles of* $\mathbf{SW}(\mathbf{Def}_Z)$*:*

$$[Y \xrightarrow{p_Y} Z] = [X \xrightarrow{p_X} Z] + [Y/X \xrightarrow{p_{Y/X}} Z],$$

*i.e.*

$$[Y] = [X] + [Y/X].$$

**4.7.2 Lemma.** $K(\mathbf{SW}(\mathbf{Def}_Z))$ *is a ring.*

*Proof.* By Corollary 3.5.11, setting $\mathbf{C} = \mathbf{Def}$ and fixing $Z \in \mathbf{Def}$. □

**4.7.3 Lemma.** *For any point* $z \in Z$*, the inclusion* $i : \{z\} \to Z$ *induces a triangulated functor*

$$i^* : \mathbf{SW}(\mathbf{Def}_Z) \to \mathbf{SW}(\mathbf{Def}_z),$$

*where* $\mathbf{SW}(\mathbf{Def}_z) = \mathbf{SW}(\mathbf{Def}_*)$*.*

*Proof.* This is a particular case of the functor $\beta^*$ for $\beta : \{z\} \to Z$.

□

**4.7.4 Theorem.** *There exists a ring isomorphism*

$$\tilde{\chi}_Z : K(\mathbf{SW}(\mathbf{Def}_Z)) \to CF(Z)$$

*given by*

$$[(X \xrightarrow{p_X} Z, m)] \mapsto (z \mapsto (-1)^m \tilde{\chi}(p_x^{-1} z)).$$

*Proof.* Since the Grothendieck group is generated by the classes of $[(X, 0)]$, it suffices to consider the case where $m = 0$. First note that $z \mapsto \tilde{\chi}(p_x^{-1} z)$ is a constructible function since there exists a relative cell decomposition of $p_x : X \to Z$ by the Trivialisation Theorem 1.1.20. Given any $[(A \xrightarrow{p_A} Z, 0)]$ in $K(\mathbf{SW}(\mathbf{Def}_Z))$, use the notation $\tilde{\chi}_Z(A)$ to denote $\tilde{\chi}_Z([(A \xrightarrow{p_A} Z, 0)])$, and consider $\tilde{\chi}_Z(A)$ restricted to a single point $z \in Z$:

$$\tilde{\chi}_Z(A)(z) = \tilde{\chi}(p_A^{-1}(z)) = \tilde{\chi}([(p_A^{-1}(z), 0)]). \tag{4.7.1}$$

Consider $i^* : \mathbf{SW}(\mathbf{Def}_Z) \to \mathbf{SW}(\mathbf{Def}_z)$ given by $i^*(A \xrightarrow{p_A} Z, 0) = (p_A^{-1}(z), 0)$ as in Lemma 4.7.3. Then, the unique induced functor

$$i^* : K(\mathbf{SW}(\mathbf{Def}_Z)) \to K(\mathbf{SW}(\mathbf{Def}_z))$$

is given by

$$i^*[(A \xrightarrow{p_A} Z, 0)] = [i^*(A \xrightarrow{p_A} Z, 0)] = [(p_A^{-1}(z), 0)]. \tag{4.7.2}$$

So, from (4.7.1) and (4.7.2), we obtain

$$\tilde{\chi}_Z(A)(z) = \tilde{\chi}([(p_A^{-1}(z), 0)]) = \tilde{\chi}(i^*[(A \xrightarrow{p_A} Z, 0)]).$$

Thus $\tilde{\chi}_Z$ is a well-defined map since $\tilde{\chi}$ is well-defined. Moreover, we observe that, since $i^*$ preserves wedge sums and smash products (Lemma 4.7.3), and since $\tilde{\chi}$ is a homomorphism (by Theorem 4.5.6), the functor $\tilde{\chi}_Z$ is also a homomorphism, i.e.

$$\tilde{\chi}_Z(A \vee_Z B) = \tilde{\chi}_Z(A) + \tilde{\chi}_Z(B)$$

and

$$\tilde{\chi}_Z(A \wedge_Z B) = \tilde{\chi}_Z(A) \cdot \tilde{\chi}_Z(B).$$

Finally we prove bijectivity. Fix a relative cell decomposition of $p_X : X \to Z$. Then consider a filtration of $X$ by unions of the preimages of the skeleta $Z_i$ of $Z$ with the copy of the base-object $Z$:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[drr] & X_1 \arrow[r] \arrow[dr] & X_2 \arrow[r] \arrow[d] & ... \arrow[r] & X_n = X, \arrow[dll] \\ & & Z & & \end{tikzcd}$$

where $X_i = p^{-1}Z_i \vee_\emptyset Z$ for $i = 0, ..., n$. Then, since

$$[X_i] = [X_{i-1}] + [C_{\mathbf{Def}_Z}(X_{i-1} \to X_i)]$$

and $[C_{\mathbf{Def}_Z}(X_{i-1} \to X_i)] = [X_i/X_{i-1}]$, we have

$$[X] = [X_0] + [X_1/X_0] + ... + [X_n/X_{n-1}].$$

Since

$$X_d/X_{d-1} \cong \Big( \bigcup_{\substack{\dim C = d \\ C \subset Z}} p^{-1}C \Big) \vee_\emptyset Z,$$

where $C$ runs over the cells in $Z$, we have

$$[X_d/X_{d-1}] = \sum_{\substack{\dim C = d \\ C \subset Z}} [p^{-1}C \vee_\emptyset Z].$$

Thus $[X]$ can be written as the sum over the cells, $C \subset Z$, of the classes of $p^{-1}C \vee_\emptyset Z$:

$$[X] = \sum_{C \subset Z} [p^{-1}C \vee_\emptyset Z].$$

For any $c \in C$, we have

$$p^{-1}C = C \times_Z p^{-1}c. \tag{4.7.3}$$

By the same argument as in the absolute case (see proof of Theorem 4.5.6), we contract skeleta in each fibre simultaneously and use result 4.7.3 to conclude that:

$$[X] = \sum_{C \subset Z} \tilde{\chi}(p^{-1}c)[C \vee_{\emptyset} Z \to Z]$$

for any $c$ in $C$, where $C \vee_{\emptyset} Z$ is an object over $Z$ in the expected way (i.e. there is an inclusion of the cell $C$ into the cell decomposition of $Z$, and $Z$ maps to $Z$ via the identity). Thus

$$\tilde{\chi}_Z[(X \xrightarrow{p} Z, 0)] = \sum_{C \subset Z} \tilde{\chi}(p^{-1}c) \cdot 1_C$$

where $c \in C$ and $1_C : Z \to \{0, 1\}$ is the indicator function of $C \subset Z$. If

$$\tilde{\chi}_Z[(X \xrightarrow{p} Z, 0)] = 0,$$

then

$$[(X \to Z, 0)] = 0.$$

Hence $\tilde{\chi}_Z$ is injective. Recall that, by Prop 1.4.2, a constructible function can be written as a linear combination of indicator functions of cells. So the indicator functions

$$1_C = \tilde{\chi}_Z(C \vee_{\emptyset} Z)$$

generate $CF(Z)$. Thus $\tilde{\chi}_Z$ is also surjective. So $\tilde{\chi}_Z$ is a bijective ring homomorphism. Therefore $\tilde{\chi}_Z$ is a ring isomorphism, as required.

□

## 4.8 Lifting the Euler calculus to $\mathbf{SW}(\mathbf{Def}_Z)$

For a fixed o-minimal expansion $\mathcal{R}$ of $\mathbb{R}$, we have shown that we can construct a family $\mathbf{SW}(\mathbf{Def}_Z)$ of tensor triangulated categories for each $Z \in \mathbf{Def}$ and that there exist adjoint functors $\beta_*$ and $\beta^*$ between these relative definable Spanier–Whitehead categories for each definable morphism $\beta \colon Z \to Z'$. We now show that these induce the operations of the Euler calculus.

Suppose $\beta : Z \to Z'$ is a morphism in $\mathbf{Def}$. Using the axiomatic approach to Spanier–Whitehead categories in Chapter 3, by Lemma 3.6.1 there exists an adjunction

$$\begin{tikzcd} \mathbf{Def}_Z \arrow[r, bend left, "\beta_*"] \arrow[r, phantom, "\perp"] & \mathbf{Def}_{Z'} \arrow[l, bend left, "\beta^*"] \end{tikzcd}$$

given by $\beta_*(\_) = \_ \vee_Z Z'$ and $\beta^*(\_) = \_ \times_{Z'} Z$. Then from Theorem 3.6.1 we know that this descends to an adjunction

$$\begin{tikzcd} \mathbf{SW}(\mathbf{Def}_Z) \arrow[rr, bend left, "\beta_*"] \arrow[rr, phantom, "\perp"] & & \mathbf{SW}(\mathbf{Def}_{Z'}) \arrow[ll, bend left, "\beta^*"] \end{tikzcd}$$

where $\beta_*$ and $\beta^*$ are triangulated and $\beta^*$ is also monoidal. This induces homomorphisms between the Grothendieck groups of $\mathbf{SW}(\mathbf{Def}_Z)$ and $\mathbf{SW}(\mathbf{Def}_{Z'})$.

**4.8.1 Theorem.** *Given $\beta : Z \to Z'$ in* **Def***, the functors*

$$\beta_* : \mathbf{Def}_Z \to \mathbf{Def}_{Z'}$$

$$\beta_*(X \xrightarrow{p} Z) = X \vee_Z Z' \xrightarrow{(\beta\circ p)\vee_Z id} Z',$$

*and*

$$\beta^* : \mathbf{Def}_{Z'} \to \mathbf{Def}_Z$$

$$\beta^*(X' \xrightarrow{p'} Z) = X' \times_{Z'} Z \xrightarrow{\pi_2} Z$$

*(where $\pi_2$ is the projection morphism onto the second factor) induce homomorphisms*

$$\begin{tikzcd} CF(Z) \arrow[rr, bend left, "\beta_*"] & & CF(Z') \arrow[ll, bend left, "\beta^*"] \end{tikzcd}$$

*where $\beta_*$ is as in Definition 1.4.5 and $\beta^*$ is as in Definition 1.4.4.*

*Proof.* Since we know that $\beta_*$ and $\beta^*$ will induce homomorphisms, it is enough to check for indicator functions. Given definable $B \subset Z'$, we note that under the identification of the Grothendieck group with the constructible functions, $[(B \vee_\emptyset Z', 0)] = 1_B$ is the indicator function of $B$. Since

$$\begin{aligned} \beta^*(B \vee_\emptyset Z') &:= (B \vee_\emptyset Z') \times_{Z'} Z \\ &= (B \times_{Z'} Z) \vee_\emptyset (Z' \times_{Z'} Z) \\ &= \beta^{-1}(B) \vee_\emptyset Z, \end{aligned}$$

we have

$$\begin{aligned} [\beta^*(B \vee_\emptyset Z', 0)] &= [(\beta^{-1}(B) \vee_\emptyset Z, 0)] \\ &= 1_{\beta^{-1}(B)} \\ &= 1_B \circ \beta \\ &= \beta^*(1_B) \end{aligned}$$

in $CF(Z)$. Hence the functor

$$\beta^* : \mathbf{SW}(\mathbf{Def}_{Z'}) \to \mathbf{SW}(\mathbf{Def}_Z)$$

induces the (ring) homomorphism

$$\beta^* : CF(Z') \to CF(Z)$$

given by composing with $\beta$. Now fix $A \subset Z$. Recall that, on the one hand, for $z' \in Z'$, we have

$$\beta_*(1_A)(z') = \chi(A \cap \beta^{-1}(z')).$$

On the other hand,

$$\begin{aligned}[\beta_*(A \vee_\emptyset Z, 0)] &= [((A \vee_\emptyset Z) \vee_Z Z', 0)] \\ &= [(A \vee_\emptyset Z', 0)] \\ &= \Big(z' \mapsto \chi(A \cap \beta^{-1}(z'))\Big).\end{aligned}$$

Hence the functor

$$\beta_* : \mathbf{SW}(\mathbf{Def}_Z) \to \mathbf{SW}(\mathbf{Def}_{Z'})$$

induces the (ring) homomorphism

$$\beta_* : CF(Z) \to CF(Z')$$

given by taking the Euler characteristic of fibres. □

This completes the construction of our homotopical categorification of the Euler calculus by definable relative versions of the Spanier-Whitehead category. An interesting future direction would be to compare this with the well-known homological categorification provided by the constructible derived category. In particular, it would be interesting to confirm that $\mathbf{SW}(\mathbf{Def}_Z)$ is not an algebraic triangulated category, and hence is not equivalent to the constructible derived category $D_c(Z; \mathbb{Q})$.

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